The Bloch Sphere

Earlier, we described the Stern-Gerlach experiment which introduced the concept of spin. As for every observable, we want to associate a Hermitian operator with the spin $\vec{S}$.

$\vec{S}$ is a form of angular momentum with three components S_x, S_y, S_z related to the Pauli spin matrices $\sigma_x, \sigma_y, \sigma_z$ by

$$\vec{S} = \frac{\hbar}{2}\vec{\sigma}$$

We postulate that these three components obey exactly the same commutation relations as the angular momentum $\vec{L}$ which we derived earlier, i.e., (see homework #3 / 2004)

$$[S_x, S_y] = i\hbar S_z$$
$$[S_y, S_z] = i\hbar S_x$$
$$[S_z, S_x] = i\hbar S_y$$

$S_x^2 = S_y^2 = S_z^2 = \frac{\hbar^2}{4}\mathbb{1}$ for the 3 Pauli matrices.

$S^2 = S_x^2 + S_y^2 + S_z^2$ and S_z commute $\rightarrow$ They are diagonalizable simultaneously. The eigenvectors are denoted as $|s, m_s\rangle$ where $s = \frac{1}{2}$, $m_s = \pm\frac{1}{2}$ for an electron

$$S^2\left|\tfrac{1}{2}, \pm\tfrac{1}{2}\right\rangle = \underbrace{\tfrac{1}{2}\left(\tfrac{1}{2}+1\right)}_{s(s+1)}\hbar^2\left|\tfrac{1}{2}, \pm\tfrac{1}{2}\right\rangle = \frac{3}{4}\hbar^2\left|\tfrac{1}{2}, \pm\tfrac{1}{2}\right\rangle$$

$$S_z\left|\tfrac{1}{2}, \pm\tfrac{1}{2}\right\rangle = \pm\tfrac{1}{2}\hbar\left|\tfrac{1}{2}, \pm\tfrac{1}{2}\right\rangle = \underbrace{S_z}_{\pm\frac{\hbar}{2}}\left|\tfrac{1}{2}, \pm\tfrac{1}{2}\right\rangle$$

Actually, $\left|\tfrac{1}{2}, +\tfrac{1}{2}\right\rangle = |0\rangle$
$\left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle = |1\rangle$

These are just different notations.

- A property of the Pauli spin matrices which was proved in a homework is

$$(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = \sigma_i \sigma_j a_i b_j$$
$$= i\,\vec{\sigma}\cdot(\vec{a}\times\vec{b}) + \vec{a}\cdot\vec{b}$$

If $\hat{n}$ is a unit vector

$$(\vec{\sigma}\cdot\vec{n})^2 = \mathbb{1}$$

which states that the square of any component of $\vec{\sigma}$ is equal to the unit 2×2 matrix. Hence, the eigenvalues of $(\vec{\sigma}\cdot\vec{n})$ are ±1 $\forall\,\vec{n}$, unit 3D-vector.

Let the eigenvectors of $(\vec{\sigma}\cdot\vec{n})$ be ξ_{n+} for $+1$ eigenvalue
ξ_{n-} for -1 eigenvalue.

We want to find the explicit expressions of ξ_{n+}, ξ_{n-}

- Consider the operator $\frac{1}{2}(\mathbb{1} \pm \vec{\sigma}\cdot\vec{n})$

Let this operator act on an arbitrary spin state χ (qubit)

Then, using the fact that $(\vec{\sigma}\cdot\vec{n})^2 = \mathbb{1}$

$$(\vec{\sigma}\cdot\vec{n})\left[\frac{1}{2}(\mathbb{1} \pm (\vec{\sigma}\cdot\vec{n}))\chi\right] = \pm\frac{1}{2}\left[(\mathbb{1} \pm (\vec{\sigma}\cdot\vec{n}))\chi\right]$$

i.e., $\forall\chi$, $\frac{1}{2}\left[\mathbb{1} \pm \vec{\sigma}\cdot\vec{n}\right]\chi$ are eigenvectors of $(\vec{\sigma}\cdot\vec{n})$ with eigenvalues ±1, respectively.

$$\frac{1}{2}(\vec{\sigma}\cdot\vec{n})\chi \pm \frac{1}{2}\underbrace{(\vec{\sigma}\cdot n)^2}_{\mathbb{1}}\chi$$

$$= \pm\frac{1}{2}\Big[\underbrace{1 \pm (\vec{\sigma}\cdot\vec{n})}\Big]\chi = (\pm 1)\underbrace{\Big[\frac{1 \pm (\vec{\sigma}\cdot\vec{n})}{2}\Big]}\chi$$

$$\frac{1}{2}\Big[(\pm)(\pm)(\vec{\sigma}\cdot\vec{n})\chi \pm \frac{1}{2}\mathbb{1}\Big]\chi$$

$$\cancel{\frac{\pm 1}{2}\left(\frac{\pm 1}{2}\right)\Big[\frac{1 \pm (\vec{\sigma}\cdot\vec{n})}{2}\Big]\chi}$$

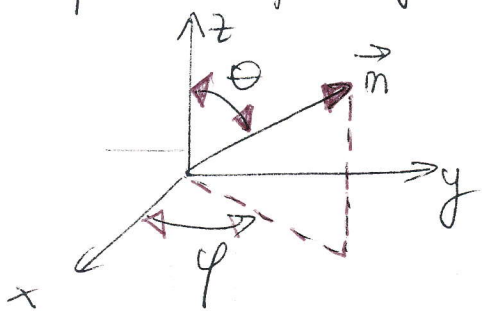
So $\Big[\frac{1 \pm \vec{\sigma}\cdot\vec{n}}{2}\Big]\chi$ is eigenstate of $(\vec{\sigma}\cdot\vec{n})$ with eigenvalue $\underline{(\pm 1)}$

Let us rewrite

$$\frac{1}{2}\left(\mathbb{1} \pm \vec{\sigma}\cdot\vec{n}\right) = \frac{1}{2}\Big[\mathbb{1} \pm \sigma_z n_z \pm \frac{1}{2}(\sigma_x + i\sigma_y)(n_x - i n_y) \pm \frac{1}{2}(\sigma_x - i\sigma_y)(n_x + i n_y)\Big]$$

which you can check easily (DO IT!).

Calling (θ, φ) the polar angles of $\vec{n}$



$$\vec{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$$

$$n_z = \cos\theta \; ; \qquad n_x \pm i n_y = \sin\theta\, e^{\pm i\varphi}$$

So,

$$\frac{1}{2}\left(\mathbb{1} \pm \vec{\sigma}\cdot\vec{n}\right) = \frac{1}{2}\Big[\mathbb{1} \pm \cos\theta\, \sigma_z \pm \sin\theta\, e^{-i\varphi}\sigma_+ \pm \sin\theta\, e^{i\varphi}\sigma_-\Big]$$

where $\sigma_+ = \sigma_x + i\sigma_y$; $\sigma_- = \sigma_x - i\sigma_y$

Taking $\begin{pmatrix}1\\0\end{pmatrix} = |0\rangle$; $\begin{pmatrix}0\\1\end{pmatrix} = |1\rangle$

$$\xi^+_{\vec{n}} = \frac{1}{2}\left(\mathbb{1} + \vec{\sigma}\cdot\vec{n}\right)|0\rangle = \cos\frac{\theta}{2}\left[\cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}\, e^{i\varphi}|1\rangle\right]$$

$$\xi^-_{\vec{n}} = \frac{1}{2}\left(\mathbb{1} - \vec{\sigma}\cdot\vec{n}\right)|1\rangle = \sin\frac{\theta}{2}\left[\sin\frac{\theta}{2}|0\rangle - \cos\frac{\theta}{2}\, e^{i\varphi}|1\rangle\right]$$

NOT normalized

This way we have generated the eigenvectors.
We still need to normalize them so that their norm is unity

we divide the first of the last two equations by $\cos\frac{\theta}{2}$ and the second by $\sin\frac{\theta}{2}$. We then have

$$\xi_{n+} = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\varphi}|1\rangle = \begin{bmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}e^{i\varphi} \end{bmatrix}$$

and

$$\xi_{n-} = \sin\frac{\theta}{2}|0\rangle - \cos\frac{\theta}{2}e^{i\varphi}|1\rangle = \begin{bmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2}e^{i\varphi} \end{bmatrix}$$

as easily checked

$$\langle\xi_{n-}|\xi_{n+}\rangle = 0 \quad \text{, i.e., } \xi_{n-}, \xi_{n+} \text{ are orthogonal and normalized}$$

The state ξ_{n+} is used by convention to represent a general qubit. Note that ξ_{n+} is defined up to an overall phase factor $e^{i\gamma}$.

We use the general ket $|\psi\rangle$ for a qubit representation

$$|\psi\rangle = \underbrace{e^{i\gamma}}_{\text{has no observable effect}}\left[\cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\varphi}|1\rangle\right]$$

The derivation above makes the appearance of the $\frac{\theta}{2}$ angle a little less mysterious. --- ONTO the Bloch sphere representation

$\langle \xi_n^+ | \sigma_x | \xi_n^+ \rangle = \sin\theta \cos\varphi$

$\langle \xi_n^+ | \sigma_y | \xi_n^+ \rangle = \sin\theta \sin\varphi$

$\langle \xi_n^+ | \sigma_z | \xi_n^+ \rangle = \cos\theta$

$\langle \xi_n^- | \sigma_x | \xi_n^- \rangle = -\sin\theta \cos\varphi$

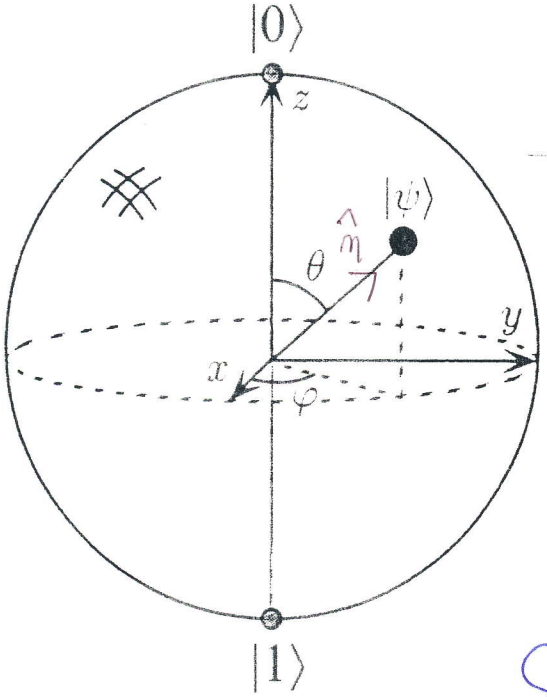
$\langle \xi_n^- | \sigma_y | \xi_n^- \rangle = -\sin\theta \sin\varphi$

$\langle \xi_n^- | \sigma_z | \xi_n^- \rangle = -\cos\theta$

BLOCH SPHERE

$(\langle\sigma_x\rangle, \langle\sigma_y\rangle, \langle\sigma_z\rangle) = \hat{n}$

$(\langle\sigma_x\rangle, \langle\sigma_y\rangle, \langle\sigma_z\rangle) = -\hat{n}$



$$\xi_n^+ = |\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle.$$

North Pole

$\theta = 0 \quad \xi_n^+ = |0\rangle \quad \forall\varphi$

$\theta = \pi \quad \xi_n^+ = |1\rangle \quad \forall\varphi$

South pole

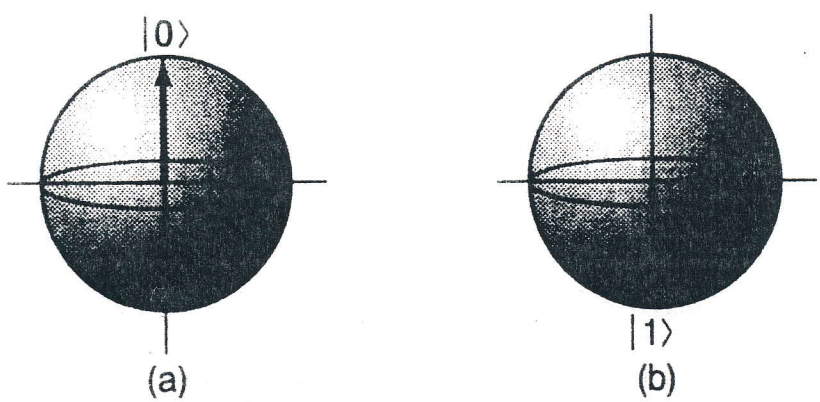
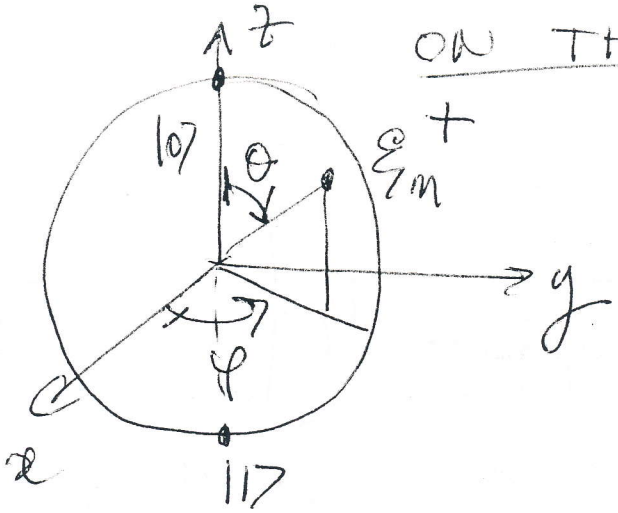


Fig. 3.6 A qubit can be pictured as a vector protruding from the center of a sphere. The south pole represents the eigenstate $|1\rangle$ and the north pole represents the eigenstate $|0\rangle$. Thus (a) shows a qubit for the binary value 0 and (b) shows a qubit for the binary value 1.

EXCURSIONS ON THE BLOCH SPHERE



$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$|0\rangle$ and $|1\rangle$ are North & South poles on Bloch sphere

$|0\rangle, |1\rangle$ are orthogonal.

$$\xi_n^+ = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}\, e^{i\varphi}|1\rangle$$

is eigenstate of $\vec{\sigma}\cdot\vec{n}$ with eigenvalue $+1$

We also have

$$\xi_n^- = \sin\frac{\theta}{2}|0\rangle - \cos\frac{\theta}{2}\, e^{i\varphi}|1\rangle$$

⚠ Careful → Same (θ, φ) in those 2 expressions

where is ξ_n^- on Bloch sphere?

$$\xi_n^- = \xi_n^+(\theta \to \pi - \theta,\ \varphi \to \varphi + \pi)$$ as easily checked

ξ_n^- is opposite to ξ_n^+ going through center of Bloch sphere

Any 2 points on Bloch sphere = intersections of straight line going through center represent orthogonal qubits.

Remarks

ξ_n^+ maps the entire Bloch sphere by varying θ from $0 \to \pi$ and $\varphi = 0 \to 2\pi$

What is 2×2 matrix η which take ξ_n^- into ξ_n^+? i.e.,

$$\xi_n^+ = \begin{bmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} e^{i\varphi} \end{bmatrix} = \eta\, \xi_n^- = \eta \begin{bmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2} e^{i\varphi} \end{bmatrix}$$

It is $\eta = \begin{bmatrix} 0 & -e^{-i\varphi} \\ e^{i\varphi} & 0 \end{bmatrix}$

Indeed $\eta\, \xi_n^- = \begin{bmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} e^{i\varphi} \end{bmatrix}$

<u>Note</u> $\eta = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & e^{i\varphi} \end{bmatrix} \begin{bmatrix} 0 & +1 \\ 1 & 0 \end{bmatrix}} \begin{bmatrix} 1 & 0 \\ 0 & e^{i(\pi-\varphi)} \end{bmatrix}$

Indeed, $\eta = \begin{bmatrix} 0 & +1 \\ e^{i\varphi} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -e^{-i\varphi} \end{bmatrix} = \begin{bmatrix} 0 & -e^{-i\varphi} \\ e^{i\varphi} & 0 \end{bmatrix}$

with

$P(\varphi) = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\varphi} \end{bmatrix}$; η = Phase shift(φ) $\underset{\text{Pauli}}{\times}$ Phase Shift$(\pi-\varphi)$

η is the true <u>qubit flip</u> matrix [η is unitary!]

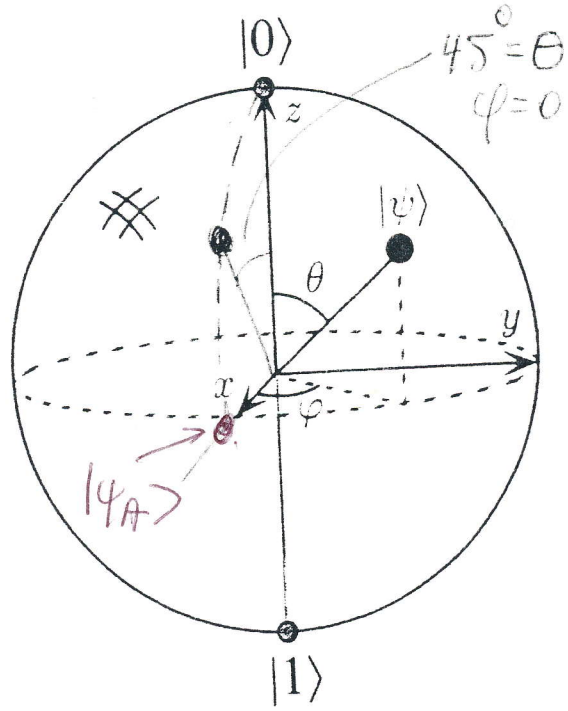
<u>Example</u>. $|\psi\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle$ (superposition state)

$$\alpha_0 = e^{i\gamma} \cos\frac{\theta}{2} \; ; \quad \alpha_1 = e^{i\gamma} e^{i\varphi} \sin\frac{\theta}{2}$$

Take $e^{i\gamma} = 1$ $\qquad \alpha_0 = \alpha_1 = \frac{1}{\sqrt{2}}$ $\qquad |\psi_A\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$

$$\alpha_0 = \cos\frac{\theta}{2} = \frac{1}{\sqrt{2}} \rightarrow \frac{\theta}{2} = 45^\circ \rightarrow \theta = \frac{\pi}{2}$$

$$\alpha_1 = e^{i\varphi} \sin\frac{\theta}{2} = \frac{1}{\sqrt{2}} \rightarrow \varphi = 0 \; ; \; \theta = \frac{\pi}{2}$$



<u>Remember</u>: A classical bit can be in one of 2 states "0" or "1". A qubit can be in a continuum of states represented as points on the Bloch sphere. The state space of a qubit contains the two "basis" or "logical" states, $|0\rangle$ and $|1\rangle$. In algorithms to be described later, many qubits will be involved. Typically, the initial state of a qubit will be one of the basis states.

As we will see in ECES 753, the Pauli matrices play an important role in error models in quantum computers and in the development of quantum error-correcting codes.

Take $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{pmatrix}\alpha\\ \beta\end{pmatrix} = e^{i\gamma}\begin{pmatrix}\cos\frac{\theta}{2}\\ e^{i\varphi}\sin\frac{\theta}{2}\end{pmatrix}$

① $\sigma_z|\psi\rangle = \sigma_z\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}\alpha\\ -\beta\end{pmatrix} = \alpha|0\rangle - \beta|1\rangle$ $\qquad \sigma_z = \text{Phase Shift}(\pi)$

So $\alpha|0\rangle + \beta|1\rangle$ has evolved to $\alpha|0\rangle - \beta|1\rangle$

This operation has changed the phase of the qubit!

We call such an operation a **"phase shift error"**. $[\varphi \to \varphi + \pi]$

② $\sigma_x|\psi\rangle = \sigma_x\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \alpha|1\rangle + \beta|0\rangle = e^{i\gamma}\begin{pmatrix}e^{i\varphi}\sin\frac{\theta}{2}\\ \cos\frac{\theta}{2}\end{pmatrix} = e^{i\gamma}e^{i\varphi}\begin{pmatrix}\sin\frac{\theta}{2}\\ e^{-i\varphi}\cos\frac{\theta}{2}\end{pmatrix}$

σ_x has caused the bits to flip, i.e., $|0\rangle \Leftrightarrow |1\rangle$

We call this operation a **"bit flip error"**. $\begin{bmatrix}\theta \to \pi - \theta\\ \varphi \to -\varphi\end{bmatrix}$

③ $\sigma_y\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}-i\beta\\ i\alpha\end{pmatrix} = i\alpha|1\rangle - i\beta|0\rangle$

i is just an overall phase factor.

σ_y corresponds to both a **"phase shift and a bit flip!"**

④ The identity matrix causes **"no error"**

Any error in a single qubit can be described by the action of a linear combination of $\mathbb{I}, \sigma_x, \sigma_y, \sigma_z$ operators.

Remember Hint

$$\boxed{q = q_0\,\mathbb{I} + \vec{q}\cdot\vec{\sigma}}$$

The $\sigma_x, \sigma_y, \sigma_z$ are unitary. Their action on the Bloch representation of a ket $|\psi\rangle$ rotate that $|\psi\rangle$ on the Bloch Sphere.

Besides $\sigma_x, \sigma_y, \sigma_z$, three other gates are typically used in building quantum circuits. Their matrix representation are also given by 2×2 matrices.

<u>Hadamard gate</u> $H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

<u>Phase Matrix</u> $S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$

(NOTE $S^2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \sigma_z$)

$S = \text{Phase Shift}\left(\frac{\pi}{2}\right)$

<u>The $\frac{\pi}{8}$ gate or T gate</u> $= \begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{i\pi}{4}} \end{bmatrix}$

Since $e^{\frac{i\pi}{4}} = \sqrt{i}$, you can think of T-gate as square root of phase gate. The $\frac{\pi}{8}$ name sounds a little funny for the T-gate since $e^{\frac{i\pi}{4}}$ appears in diagonal.

Actually, up to a phase factor

$$T = e^{\frac{i\pi}{8}} \begin{bmatrix} e^{-\frac{i\pi}{8}} & 0 \\ 0 & e^{+\frac{i\pi}{8}} \end{bmatrix}$$

$[T^2 = S]$

$$\text{Phase Shift}(\varphi) = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\varphi} \end{bmatrix}$$

We can arrive at $|\psi\rangle = \frac{1}{\sqrt{2}}\{|0\rangle + |1\rangle\}$ from state $|0\rangle$ by rotating the qubit $|0\rangle$ by 90° around the y-axis.

Actually, consider the matrix

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

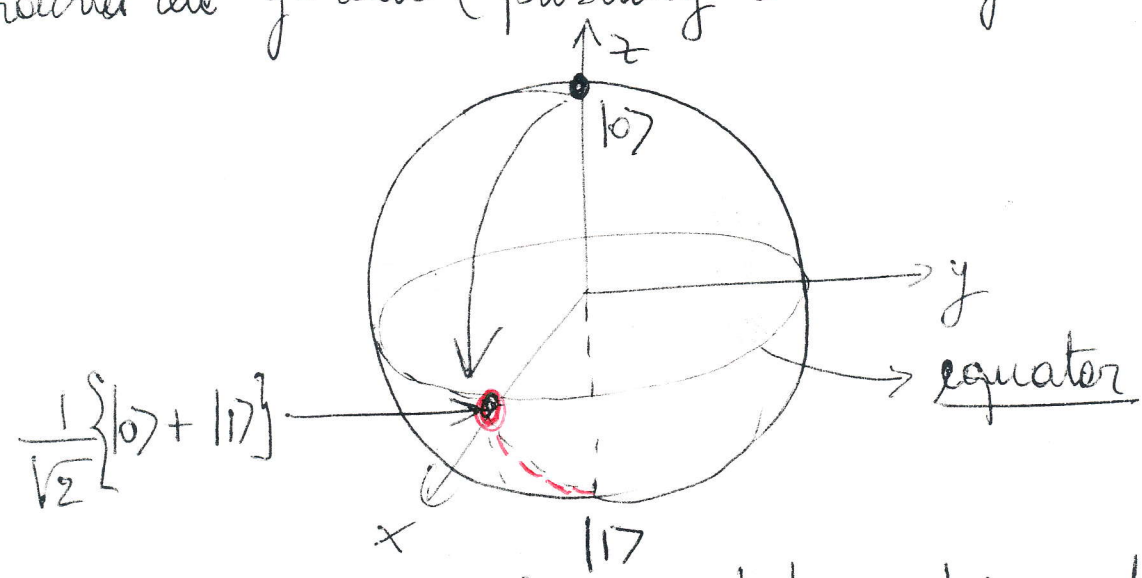
which will appear many times later in the design of quantum circuits. It is referred to as the <u>Hadamard gate</u>.

Acting on $|0\rangle$, H gives: $H|0\rangle = \frac{1}{\sqrt{2}}\{|0\rangle + |1\rangle\}$

Acting on $|1\rangle$, H gives: $H|1\rangle = \frac{1}{\sqrt{2}}\{|0\rangle - |1\rangle\}$

The latter state is also a ket on the Bloch sphere with $\gamma = 0$, $\varphi = \pi$, $\theta = \frac{\pi}{2}$ (see below)

H seems to be performing the equivalent of a rotation around the y-axis (pushing z on x) by a 90° angle.



$\frac{1}{\sqrt{2}}\{|0\rangle + |1\rangle\}$ is halfway between $|0\rangle$ and $|1\rangle$.

For that reason, H is sometime referred to as the square root of NOT gate.

This is not really a good denomination. Indeed, the NOT gate is one for which

$$\text{NOT}\,|0\rangle = |1\rangle$$
$$\text{NOT}\,|1\rangle = |0\rangle$$

In the basis $|0\rangle, |1\rangle$, the matrix representation of the NOT gate is therefore

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

which is the Pauli matrix σ_x or X.

Clearly, with our interpretation of a square root, we should have

$$H^2 = X$$

But, as checked easily $H^2 = \mathbb{I}$ which is not X!

So, referring to the Hadamard matrix (gate) as the square root of NOT gate is not really a good denomination!

Actually, the Hadamard gate performs a rotation on the Bloch sphere around the y-axis by 90° (pushing z towards x using shortest path), followed by a reflection through the $\hat{x}$-$\hat{y}$ plane. This is the geometrical interpretation of the action of H on <u>ANY</u> ket $|\psi\rangle$ on the Bloch sphere.

<u>Illustration</u>:

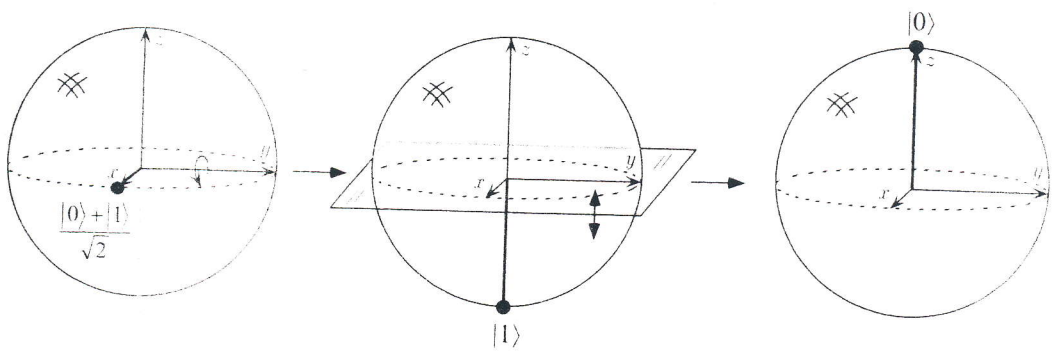
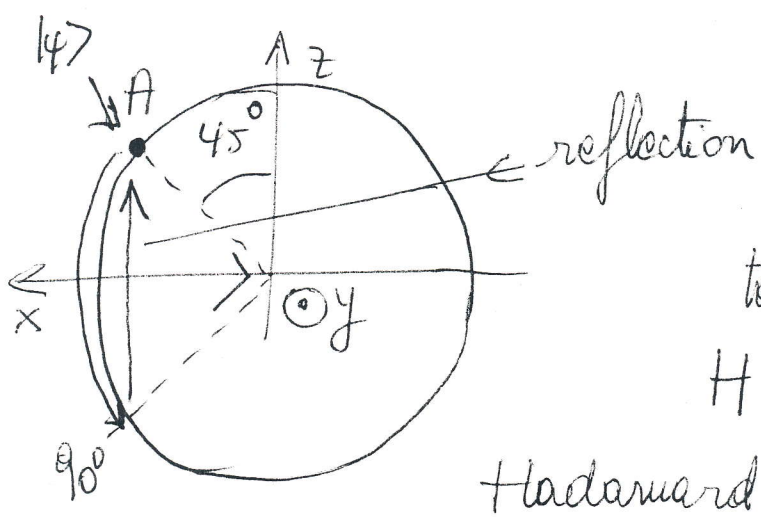


Figure 1.4. Visualization of the Hadamard gate on the Bloch sphere, acting on the input state $(|0\rangle + |1\rangle)/\sqrt{2}$.

<u>Another example</u>



So if $|\psi\rangle$ = point A on the sphere we should be able to show that $H|\psi_A\rangle = |\psi_A\rangle$, i.e., the Hadamard gate has no effect on $|\psi_A\rangle$ in that case

α_0, α_1? $\quad |\psi_A\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$

Take $\gamma = 0$; $\varphi = 0$, $\theta = \frac{\pi}{4}$

$$|\psi_A\rangle = \cos\frac{\pi}{8}|0\rangle + \sin\frac{\pi}{8}|1\rangle = \begin{bmatrix} \cos\frac{\pi}{8} \\ \sin\frac{\pi}{8} \end{bmatrix}$$

$$H|\psi_A\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \cos\frac{\pi}{8} \\ \sin\frac{\pi}{8} \end{bmatrix}$$

$$H|\psi_A\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}}\left(\cos\frac{\pi}{8} + \sin\frac{\pi}{8}\right) \\ \frac{1}{\sqrt{2}}\left(\cos\frac{\pi}{8} - \sin\frac{\pi}{8}\right) \end{bmatrix}$$

But $\frac{1}{\sqrt{2}} = \cos\frac{\pi}{4} = \sin\frac{\pi}{4}$

Hence $$H|\psi_A\rangle = \begin{bmatrix} \cos\frac{\pi}{4}\cos\frac{\pi}{8} + \sin\frac{\pi}{4}\sin\frac{\pi}{8} \\ \sin\frac{\pi}{4}\cos\frac{\pi}{8} - \cos\frac{\pi}{4}\sin\frac{\pi}{8} \end{bmatrix}$$

We use $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$
$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

So $$H|\psi_A\rangle = \begin{bmatrix} \cos\left(\frac{\pi}{4} - \frac{\pi}{8}\right) \\ \sin\left(\frac{\pi}{4} - \frac{\pi}{8}\right) \end{bmatrix} = \begin{bmatrix} \cos\frac{\pi}{8} \\ \sin\frac{\pi}{8} \end{bmatrix} = |\psi_A\rangle \,!$$

In other words, the Hadamard gate does not do much on your spinor if you are too close to the tropics on the Bloch Sphere!

What would the Hadamard gate be good for if we can find a physical system to implement it?

Take $|\psi\rangle = |0\rangle$ initial state $= \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$H|0\rangle \text{ is } \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

If we perform a measurement on $H|0\rangle$, we find that the probability to find the system in state $|0\rangle$ is (remember Postulate 3 on measurements)

$$|\langle 0| H|0\rangle|^2 = \frac{1}{2}$$

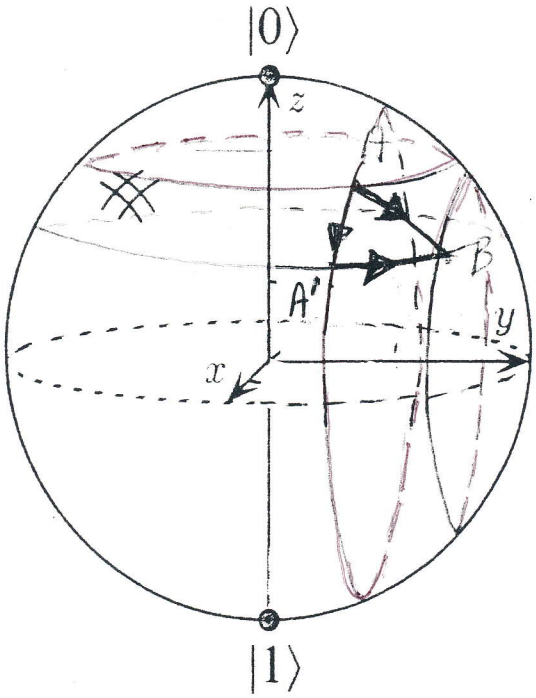
and to be in state $|1\rangle$

$$|\langle 1| H|0\rangle|^2 = \frac{1}{2}$$

equal probability

So, with the Hadamard gate, we are able to generate a perfect random number generator. A single computation with a single gate yields a perfectly random bit! This is much better than a classical algorithm which requires more than one gate and many computational steps to generate a random number. Actually, the number generated classically is never truly random. It is called pseudo-random.

Take two points A & B on the Bloch sphere below



To go from A to B, we could do a succession of two rotations, the first one being around the y-axis to go from $A \to A'$, the second one being a rotation around the z-axis from $A' \to B$.

Could the order of the rotations be reversed? If we first rotate around z, we will not be able to find a rotation around y to bring A' onto B. This is because B is closer to the equator (plane x,y) than A.

If B was closer to the pole (north), we could go from $A \to B$ via a point A' by first rotating about the z-axis followed by a rotation around the y-axis.

Rotations on the Bloch sphere play a fundamental role in describing the motion of qubits. We study rotations in more details next

Section 4.3 Rotation operations on the Bloch sphere

What are the matrices describing rotations about the x, y, and z axes?

We first prove the following identity: If θ is real and if a matrix A is such that $A^2 = \mathbb{1}$, then

$$e^{i\theta A} = \cos\theta\, \mathbb{1} + i \sin\theta\, A$$

From the Taylor series expansions

$$e^x = \sum_{k=0}^{\infty} x^k / k!$$

$$\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

$$\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$

we get

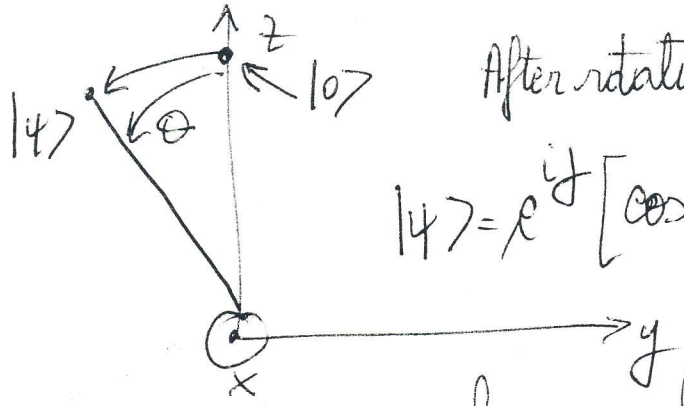
$$e^{i\theta A} = \mathbb{1} + (i\theta) A + \frac{(i\theta)^2}{2!} A^2 + \frac{(i\theta)^3}{3!} A^3 + \cdots$$

$$\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} \cdots + (-1)^k \frac{(\theta)^{2k}}{2k!} + \cdots\right) \mathbb{1}$$

$$+ i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots + (-1)^k \frac{(i\theta)^{2k+1}}{(2k+1)!} + \cdots\right) A$$

So, indeed $\boxed{e^{i\theta A} = \cos\theta\, \mathbb{1} + i \sin\theta\, A}$

Rotation Matrix by θ around x-axis? $R_x(\theta)$?



After rotation $|0\rangle \to |4\rangle$

$$|4\rangle = e^{i\gamma}\left[\cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle\right]$$

$$|4\rangle \overset{?}{=} R_x(\theta)|0\rangle$$

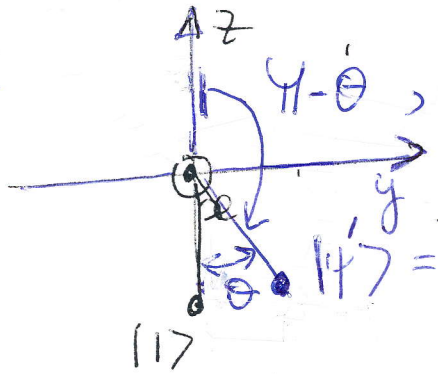
Take $\gamma = 0$ for simplicity

for $|4\rangle$, $\varphi = -\frac{\pi}{2}$

$$\text{So } |4\rangle = \cos\frac{\theta}{2}|0\rangle + e^{-i\frac{\pi}{2}}\sin\frac{\theta}{2}|1\rangle = \begin{bmatrix}\cos\frac{\theta}{2}\\ e^{-i\frac{\pi}{2}}\sin\frac{\theta}{2}\end{bmatrix}$$

$$|4\rangle = \begin{bmatrix}\cos\frac{\theta}{2}\\ -i\sin\frac{\theta}{2}\end{bmatrix}$$

Starting with $|1\rangle$



$\pi - \theta$, $\varphi = \frac{\pi}{2}$

$$|4'\rangle = e^{i\gamma}\left[\cos\left(\frac{\pi-\theta}{2}\right)|0\rangle + e^{i\frac{\pi}{2}}\sin\left(\frac{\pi-\theta}{2}\right)|1\rangle\right]$$

$$|4'\rangle = R_x(\theta)\,|1\rangle = \cos\left(\frac{\pi-\theta}{2}\right)|0\rangle + e^{+i\frac{\pi}{2}}\sin\left(\frac{\pi-\theta}{2}\right)|1\rangle$$

$$= +\sin\frac{\theta}{2}|0\rangle + i\cos\frac{\theta}{2}|1\rangle$$

$$= \begin{bmatrix}+\sin\frac{\theta}{2}\\ +i\cos\frac{\theta}{2}\end{bmatrix} = (+i)\begin{bmatrix}-i\sin\frac{\theta}{2}\\ \cos\frac{\theta}{2}\end{bmatrix}$$

phase factor can be absorbed in $e^{i\gamma}$

So, we can write explicitely

$$R_x(\theta) = \begin{pmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$ is unitary

This matrix can be obtained from σ_x as follows

$$R_x(\theta) = e^{-i\left(\frac{\theta}{2}\right)\sigma_x}$$

using the identity $e^{i\theta A} = \cos\theta + i\sin\theta\, A$

with $\theta \to -\frac{\theta}{2}$ $\quad A = \sigma_x$; $A^2 = \sigma_x^2 = \mathbb{I}$

So

$$R_x(\theta) = \left(\cos\frac{\theta}{2}\right)\mathbb{I} - i\left(\sin\frac{\theta}{2}\right)\sigma_x \quad ; \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$R_x(\theta) = \begin{bmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}$$

Notice that $R_x(\theta)$ is unitary!

Similarly, it can be shown

$$R_y(\theta) = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$ for rotation around y-axis

$$R_y(\theta) = \cos\left(\frac{\theta}{2}\right)\mathbb{I} - i\sin\left(\frac{\theta}{2}\right)\sigma_y$$

$$R_z(\theta) = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}$$ for rotation around z-axis.

$$R_z(\theta) = \cos\left(\frac{\theta}{2}\right)\mathbb{I} - i\sin\left(\frac{\theta}{2}\right)\sigma_z$$

Next, we investigate the <u>composition of rotations</u>

The rotation matrices around the x, y, z axes found above satisfy the following relations

$$(1)\ R_y(\theta_1)\,R_y(\theta_2) = R_y(\theta_1+\theta_2)$$
$$(2)\ R_z(\theta_1)\,R_z(\theta_2) = R_z(\theta_1+\theta_2)$$
$$(3)\ R_x(\theta_1)\,R_x(\theta_2) = R_x(\theta_1+\theta_2)$$

<u>we prove the first one</u>

$$R_y(\theta_1)\,R_y(\theta_2) = \begin{pmatrix} \cos(\theta_1/2) & -\sin(\theta_1/2) \\ \sin(\theta_1/2) & \cos(\theta_1/2) \end{pmatrix} \begin{pmatrix} \cos(\theta_2/2) & -\sin(\theta_2/2) \\ \sin(\theta_2/2) & \cos(\theta_1/2 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\left(\frac{\theta_1+\theta_2}{2}\right) & -\sin\left(\frac{\theta_1+\theta_2}{2}\right) \\ \sin\left(\frac{\theta_1+\theta_2}{2}\right) & \cos\left(\frac{\theta_1+\theta_2}{2}\right) \end{pmatrix} = R_y(\theta_1+\theta_2)$$

where we have used the following trigonometric identities

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$
$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

The two remaining relations ((2) and (3) above) can be proved using the same identities.

$\boxed{\text{Exercise } 4.7}$ p175 in Nielsen and Chuang

Show that $XYX = -Y$. X, Y are the Pauli matrices

Use this result to show that

$$X R_y(\theta) X = R_y(-\theta)$$

Identities like this one will be very useful later when reducing the complexity of quantum circuits.

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \rightarrow XY = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$\rightarrow XYX = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} = -Y$$

$$X R_y(\theta) X = X \left[\cos\frac{\theta}{2} \mathbb{I} - i \sin\frac{\theta}{2} Y \right] X$$

$$= \cos\frac{\theta}{2} \underset{\updownarrow}{X^2} - i \sin\frac{\theta}{2} XYX$$

$$= \cos\frac{\theta}{2} \mathbb{I} + i \sin\frac{\theta}{2} Y$$

$$= R_y(-\theta)$$

Exercise 4.15 Composition of single qubit operations

p. 177

The Bloch sphere representation gives a nice way to visualize the effect of composing 2 rotations.

Prove that if a rotation through an angle β_1 about the axis $\hat{n}_1$ is followed by a rotation through an angle β_2 about an axis $\hat{n}_2$, then the overall rotation is through an angle β_{12} about an axis $\hat{n}_{12}$ such that

$$c_{12} = c_1 c_2 - s_1 s_2 (\hat{n}_1 \cdot \hat{n}_2)$$

$$s_{12}\, \hat{n}_{12} = s_1 c_2\, \hat{n}_1 + c_1 s_2\, \hat{n}_2 - s_1 s_2 (\hat{n}_2 \times \hat{n}_1)$$

where we have defined

$$c_i = \cos\left(\frac{\beta_i}{2}\right)$$

$$s_i = \sin\left(\frac{\beta_i}{2}\right)$$

$$c_{12} = \cos\left(\frac{\beta_{12}}{2}\right)$$

$$s_{12} = \sin\left(\frac{\beta_{12}}{2}\right)$$

Remember

$$R_{\hat{n}}(\theta_{12}) = e^{-i\frac{\theta_{12}}{2}(\hat{n}\cdot\vec{\sigma})}$$

$$\parallel$$

$$\cos\left(\frac{\theta_{12}}{2}\right) \mathbb{I} - i \sin\left(\frac{\theta_{12}}{2}\right) \vec{\sigma}\cdot\hat{n}_{12}$$

Exercises

$$R_{\hat{n}}(\theta) \equiv e^{-\frac{i\theta}{2}\hat{n}\cdot\hat{\sigma}}$$

$$= \cos\left(\frac{\theta}{2}\right)\mathbb{1} - i\sin\left(\frac{\theta}{2}\right)\left(n_x\sigma_x + n_y\sigma_y + n_z\sigma_z\right)$$

$$R_{\hat{n}_2}(\beta_2)R_{\hat{n}_1}(\beta_1)$$

$$= \left[\cos\left(\frac{\beta_1}{2}\right)\mathbb{1} - i\sin\left(\frac{\beta_1}{2}\right)\left(n_x^1\sigma_x + n_y^1\sigma_y + n_z^1\sigma_z\right)\right]$$
$$\left[\cos\left(\frac{\beta_2}{2}\right)\mathbb{1} - i\sin\left(\frac{\beta_2}{2}\right)\left(n_x^2\sigma_x + n_y^2\sigma_y + n_z^2\sigma_z\right)\right]$$

<u>Use</u> $\sigma_j\sigma_k = \delta_{jk}\mathbb{1} + i\sum_{l=1}^{3}\varepsilon_{jkl}\sigma_l$

$$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = +1$$
$$\varepsilon_{132} = \varepsilon_{213} = \varepsilon_{321} = -1$$

$$\sigma_x\sigma_y = i\left(\varepsilon_{xy1}\sigma_1 + \varepsilon_{xy2}\sigma_2 + \varepsilon_{xy3}\sigma_3\right)$$
$$\sigma_x\sigma_y = i\,\varepsilon_{xyz}\sigma_z = i\sigma_z$$

$$= \cos\frac{\beta_1}{2}\cos\frac{\beta_2}{2} - i\cos\frac{\beta_1}{2}\sin\frac{\beta_2}{2}\left(n_x^2\sigma_x + n_y^2\sigma_y + n_z^2\sigma_z\right)$$
$$- i\cos\frac{\beta_2}{2}\sin\frac{\beta_1}{2}\left(n_x^1\sigma_x + n_y^1\sigma_y + n_z^1\sigma_z\right)$$
$$- \sin\frac{\beta_1}{2}\sin\frac{\beta_2}{2}\Big(n_x^1 n_x^2\sigma_x^2 + n_y^1 n_y^2\sigma_y^2 + n_z^1 n_z^2\sigma_z^2$$
$$+ n_x^1 n_y^2\sigma_x\sigma_y + n_x^1 n_z^2\sigma_x\sigma_z$$
$$+ n_y^1 n_x^2\sigma_y\sigma_x + n_y^1 n_z^2\sigma_y\sigma_z$$
$$+ n_z^1 n_x^2\sigma_z\sigma_x + n_z^1 n_y^2\sigma_z\sigma_y\Big)$$

So $R_{\hat{n}_2}(B_2)\, R_{\hat{n}_1}(B_1)$

$$= \left(\cos\left(\frac{B_1}{2}\right)\cos\left(\frac{B_2}{2}\right) - \sin\left(\frac{B_1}{2}\right)\sin\left(\frac{B_2}{2}\right)\hat{n}_1\cdot\hat{n}_2\right)\mathbb{1}$$

$$\boxed{\begin{aligned} &- i\cos\left(\frac{B_1}{2}\right)\sin\left(\frac{B_2}{2}\right)\left(n_x^2\sigma_x + n_y^2\sigma_y + n_z^2\sigma_z\right) \\ &- i\cos\left(\frac{B_2}{2}\right)\sin\left(\frac{B_1}{2}\right)\left(n_x^1\sigma_x + n_y^1\sigma_y + n_z^1\sigma_z\right)\end{aligned}}$$

$$\boxed{\begin{aligned} &- \sin\left(\frac{B_1}{2}\right)\sin\left(\frac{B_2}{2}\right)\Big[\, i\left(n_y^1 n_z^2 - n_z^1 n_y^2\right)\sigma_x \\ &\qquad + i\left(n_z^1 n_x^2 - n_x^1 n_z^2\right)\sigma_y \\ &\qquad + i\left(n_x^1 n_y^2 - n_y^1 n_x^2\right)\sigma_z \Big]\end{aligned}}$$

$$- i\, c_1 s_2\, \hat{n}_2\cdot\vec{\sigma} - i\, c_2 s_1\, \hat{n}_1\cdot\vec{\sigma}$$

with $c_1 = \cos\left(\frac{B_1}{2}\right)$, $c_2 = \cos\left(\frac{B_2}{2}\right)$, $s_1 = \sin\frac{B_1}{2}$, $s_2 = \sin\left(\frac{B_2}{2}\right)$

$$- i s_1 s_2 \left\{ \begin{aligned} &\left(n_y^1 n_z^2 - n_z^1 n_y^2\right)\hat{i}\cdot\vec{\sigma} \\ &+ \left(n_z^1 n_x^2 - n_z^2 n_x^1\right)\hat{j}\cdot\vec{\sigma} \\ &+ \left(n_x^1 n_y^2 - n_y^1 n_x^2\right)\hat{k}\cdot\vec{\sigma}\end{aligned}\right\} = - i s_1 s_2 \cdot \underbrace{\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ n_x^1 & n_y^1 & n_z^1 \\ n_x^2 & n_y^2 & n_z^2 \end{vmatrix}}_{\hat{n}_1\times\hat{n}_2}\cdot\vec{\sigma}$$

So $\boxed{R_{\hat{n}_2}(B_2)\, R_{\hat{n}_1}(B_1) = c_{12}\,\mathbb{1} - i\, s_{12}\, \hat{n}_{12}\cdot\vec{\sigma}}$

where $c_{12} = c_1 c_2 - s_1 s_2\, \hat{n}_1\cdot\hat{n}_2$

$\hat{n}_{12}\, s_{12} = s_1 c_2\, \hat{n}_1 + s_2 c_1\, \hat{n}_2 - s_1 s_2\, \hat{n}_2\times\hat{n}_1$