1 Lempel’s Construction

Let \([s] = (..., s_{-2}, s_{-1}, s_0, s_1, s_2, ...)\) denote a bi-infinite sequence over values in \(\mathbb{Z}_c\). That is, for all \(i\), \(s_i \in \mathbb{Z}_c\).

We assume that \([s]\) is periodic with a least period of \(l\).

We define the weight, \(wt([s])\) of the sequence as:

\[
wt([s]) = \sum_{i=0}^{l-1} s_i
\]

**Property 1.1.** Let \([s]\) be a De Bruijn sequence of order \(n\) over values in \(\mathbb{Z}_c\). If \(n \geq 2\), then the weight of \([s]\) is 0 (mod \(c\)). That is,

\[
wt([s]) = \sum_{i=0}^{l-1} s_i \equiv 0 \pmod{c}
\]

**Proof.** The least period of the sequence, \(l\), is \(c^n\). Each value \(a \in \mathbb{Z}_c\) appears \(c^{n-1}\) times in a single period. Thus,

\[
\sum_{i=0}^{l-1} s_i = \sum_{a=0}^{c-1} ae^{n-1} \equiv 0 \pmod{c}
\]

**Definition 1.2.** The difference operator, \(D\), maps a sequence, \([s] = (..., s_{-2}, s_{-1}, s_0, s_1, s_2, ...)\), over \(\mathbb{Z}_c\), to a sequence \([t] = (..., t_{-2}, t_{-1}, t_0, t_1, t_2, ...)\) such that \(t_i = s_{i+1} - s_i\), or \(s_{i+1} = s_i + t_i\).

Given an arbitrary value \(b \in \mathbb{Z}_c\) and a sequence \([t]\), one may compute the terms of a sequence \([s]\). That is,

\[
s_i = \begin{cases} 
  b & \text{if } i = 0, \\
  b + \sum_{j=0}^{i-1} t_j & \text{if } i > 0, \text{ or} \\
  b0 - \sum_{j=i}^{i-1} t_j & \text{if } i < 0.
\end{cases}
\]

From this, we can define an inverse difference operator, \(D_b^{-1}\).