15.1.2 General Two-Pole Active Filter

Consider Figure 15.5 with admittances Y_1 through Y_4 and an ideal voltage follower. We will derive the transfer function for the general network and will then apply specific admittances to obtain particular filter characteristics.

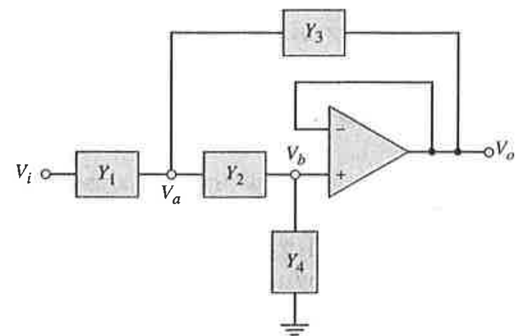

Figure 15.5 General two-pole active filter

A KCL equation at node V_a yields

$$(V_i - V_a)Y_1 = (V_a - V_b)Y_2 + (V_a - V_o)Y_3 \tag{15.3}$$

A KCL equation at node V_b produces

$$(V_a - V_b)Y_2 = V_bY_4 \tag{15.4}$$

From the voltage follower characteristics, we have $V_b = V_o$. Therefore, Equation (15.4) becomes

$$V_a = V_b\left(\frac{Y_2 + Y_4}{Y_2}\right) = V_o\left(\frac{Y_2 + Y_4}{Y_2}\right) \tag{15.5}$$

Substituting Equation (15.5) into (15.3) and again noting that $V_b = V_o$, we have

$$\begin{aligned} V_iY_1 + V_o(Y_2 + Y_3) &= V_a(Y_1 + Y_2 + Y_3) \\ &= V_o\left(\frac{Y_2 + Y_4}{Y_2}\right)(Y_1 + Y_2 + Y_3) \end{aligned} \tag{15.6}$$

Multiplying Equation (15.6) by Y_2 and rearranging terms, we get the following expression for the transfer function:

$$T(s) = \frac{V_o(s)}{V_i(s)} = \frac{Y_1Y_2}{Y_1Y_2 + Y_4(Y_1 + Y_2 + Y_3)} \tag{15.7}$$

To obtain a low-pass filter, both Y_1 and Y_2 must be conductances, allowing the signal to pass into the voltage follower at low frequencies. If element Y_4 is a capacitor, then the output rolls off at high frequencies.

To produce a two-pole function, element Y_3 must also be a capacitor. On the other hand, if elements Y_1 and Y_2 are capacitors, then the signal will be blocked at low frequencies but will be passed into the voltage follower at high frequencies, resulting in a high-pass filter. Therefore, admittances Y_3 and Y_4 must both be conductances to produce a two-pole high-pass transfer function.

15.1.3 Two-Pole Low-Pass Butterworth Filter

To form a low-pass filter, we set $Y_1 = G_1 = 1/R_1$, $Y_2 = G_2 = 1/R_2$, $Y_3 = sC_3$, and $Y_4 = sC_4$, as shown in Figure 15.6. The transfer function, from Equation (15.7), becomes

$$T(s) = \frac{V_o(s)}{V_i(s)} = \frac{G_1G_2}{G_1G_2 + sC_4(G_1 + G_2 + sC_3)} \quad (15.8)$$

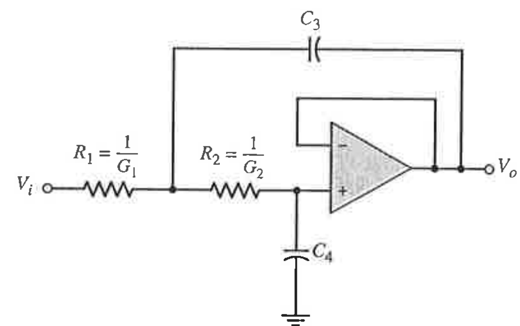

Figure 15.6 General two-pole low-pass filter

At zero frequency, $s = j\omega = 0$ and the transfer function is

$$T(s = 0) = \frac{G_1G_2}{G_1G_2} = 1 \quad (15.9)$$

In the high-frequency limit, $s = j\omega \rightarrow \infty$ and the transfer function approaches zero. This circuit therefore acts as a low-pass filter.

A **Butterworth filter** is a **maximally flat magnitude filter**. The transfer function is designed such that the magnitude of the transfer function is as flat as possible within the passband of the filter. This objective is achieved by taking the derivatives of the transfer function with respect to frequency and setting as many as possible equal to zero at the center of the passband, which is at zero frequency for the low-pass filter.

Let $G_1 = G_2 \equiv G = 1/R$. The transfer function is then

$$T(s) = \frac{\frac{1}{R^2}}{\frac{1}{R^2} + sC_4\left(\frac{2}{R} + sC_3\right)} = \frac{1}{1 + sRC_4(2 + sRC_3)} \quad (15.10)$$

We define time constants at $\tau_3 = RC_3$ and $\tau_4 = RC_4$. If we then set $s = j\omega$, we obtain

$$T(j\omega) = \frac{1}{1 + j\omega\tau_4(2 + j\omega\tau_3)} = \frac{1}{(1 - \omega^2\tau_3\tau_4) + j(2\omega\tau_4)} \quad (15.11)$$

The magnitude of the transfer function is therefore

$$|T(j\omega)| = \left[(1 - \omega^2\tau_3\tau_4)^2 + (2\omega\tau_4)^2\right]^{-1/2} \quad (15.12)$$

For a maximally flat filter (that is, a filter with a minimum rate of change), which defines a Butterworth filter, we set

$$\left.\frac{d|T|}{d\omega}\right|_{\omega=0} = 0 \tag{15.13}$$

Taking the derivative, we find

$$\frac{d|T|}{d\omega} = -\frac{1}{2}[(1-\omega^2\tau_3\tau_4)^2 + (2\omega\tau_4)^2]^{-3/2}[-4\omega\tau_3\tau_4(1-\omega^2\tau_3\tau_4) + 8\omega\tau_4^2] \tag{15.14}$$

Setting the derivative equal to zero at $\omega = 0$ yields

$$\begin{aligned}\left.\frac{d|T|}{d\omega}\right|_{\omega=0} &= \left[-4\omega\tau_3\tau_4(1-\omega^2\tau_3\tau_4) + 8\omega\tau_4^2\right] \\ &= 4\omega\tau_4\left[-\tau_3(1-\omega^2\tau_3\tau_4) + 2\tau_4\right]\end{aligned} \tag{15.15}$$

Equation (15.15) is satisfied when $2\tau_4 = \tau_3$, or

$$C_3 = 2C_4 \tag{15.16}$$

For this condition, the transfer magnitude is, from Equation (15.12),

$$|T| = \frac{1}{\left[1 + 4(\omega\tau_4)^4\right]^{1/2}} \tag{15.17}$$

The 3 dB, or cutoff, frequency occurs when $|T| = 1/\sqrt{2}$, or when $4(\omega_{3\,\text{dB}}\tau_4)^4 = 1$. We then find that

$$\omega_{3\,\text{dB}} = 2\pi f_{3\,\text{dB}} = \frac{1}{\tau_4\sqrt{2}} = \frac{1}{\sqrt{2}RC_4} \tag{15.18}$$

In general, we can write the cutoff frequency in the form

$$\omega_{3\,\text{dB}} = \frac{1}{RC} \tag{15.19}$$

Finally, comparing Equations (15.19), (15.18), and (15.16) yields

$$C_4 = 0.707C \tag{15.20(a)}$$

and

$$C_3 = 1.414C \tag{15.20(b)}$$

The two-pole low-pass Butterworth filter is shown in Figure 15.7(a). The Bode plot of the transfer function magnitude is shown in Figure 15.7(b). From Equation (15.17), the magnitude of the voltage transfer function for the two-pole low-pass Butterworth filter can be written as

$$|T| = \frac{1}{\sqrt{1 + \left(\dfrac{f}{f_{3\,\text{dB}}}\right)^4}} \tag{15.21}$$

Equation (15.15) shows that the derivative of the voltage transfer function magnitude at $\omega = 0$ is zero even without setting $2\tau_4 = \tau_3$. However, the added condition of $2\tau_4 = \tau_3$ produces the maximally flat transfer characteristics of the Butterworth filter.

For a maximally flat filter (that is, a filter with a minimum rate of change), which defines a Butterworth filter, we set

$$\left.\frac{d|T|}{d\omega}\right|_{\omega=0} = 0 \tag{15.13}$$

Taking the derivative, we find

$$\frac{d|T|}{d\omega} = -\frac{1}{2}[(1-\omega^2\tau_3\tau_4)^2 + (2\omega\tau_4)^2]^{-3/2}[-4\omega\tau_3\tau_4(1-\omega^2\tau_3\tau_4) + 8\omega\tau_4^2] \tag{15.14}$$

Setting the derivative equal to zero at $\omega = 0$ yields

$$\begin{aligned}\left.\frac{d|T|}{d\omega}\right|_{\omega=0} &= \left[-4\omega\tau_3\tau_4(1-\omega^2\tau_3\tau_4) + 8\omega\tau_4^2\right] \\ &= 4\omega\tau_4\left[-\tau_3(1-\omega^2\tau_3\tau_4) + 2\tau_4\right]\end{aligned} \tag{15.15}$$

Equation (15.15) is satisfied when $2\tau_4 = \tau_3$, or

$$C_3 = 2C_4 \tag{15.16}$$

For this condition, the transfer magnitude is, from Equation (15.12),

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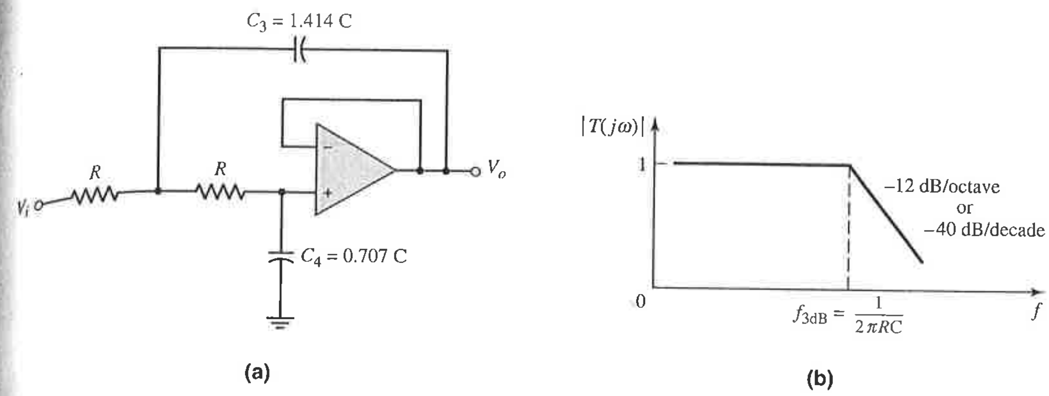

Figure 15.7 (a) Two-pole low-pass Butterworth filter and (b) Bode plot, transfer function magnitude

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Design Example 15.1 **Objective:** Design a two-pole low-pass Butterworth filter for an audio amplifier application.

Consider the circuit shown in Figure 15.7(a). Design the circuit such that the bandwidth is 20 kHz.

Solution: From Equation (15.19), we have

$$f_{3\,dB} = \frac{1}{2\pi RC}$$

or

$$RC = \frac{1}{2\pi f_{3\,dB}} = \frac{1}{2\pi(20 \times 10^3)} = 7.96 \times 10^{-6}$$

If we let $R = 100\,k\Omega$, then $C = 79.6\,pF$, which means tht $C_3 = 1.414C = 113\,pF$ and $C_4 = 0.707C = 56.3\,pF$.

Comment: These resistance and capacitance values are generally too large to be fabricated conveniently on an IC. Instead, discrete resistors and capacitors, in conjunction with the IC op-amp, would need to be used.

Computer Simulation Verification: Figure 15.8(a) shows the circuit used in the computer simulation. A standard LM324 op-amp is used. A 1 V sinusoidal input signal is applied. Figure 15.8(b) shows the output signal as a function of frequency. The 3 dB frequency, the frequency at which the output signal is 0.707 V, is 20 kHz, as designed. The slope of the rolloff at high frequency is also −12 dB/octave, as predicted from theory.

15.1.4 Two-Pole High-Pass Butterworth Filter

To form a high-pass filter, the resistors and capacitors are interchanged from those in the low-pass filter. A two-pole high-pass Butterworth filter is shown in Figure 15.9(a). The analysis proceeds exactly the same as in the last section,

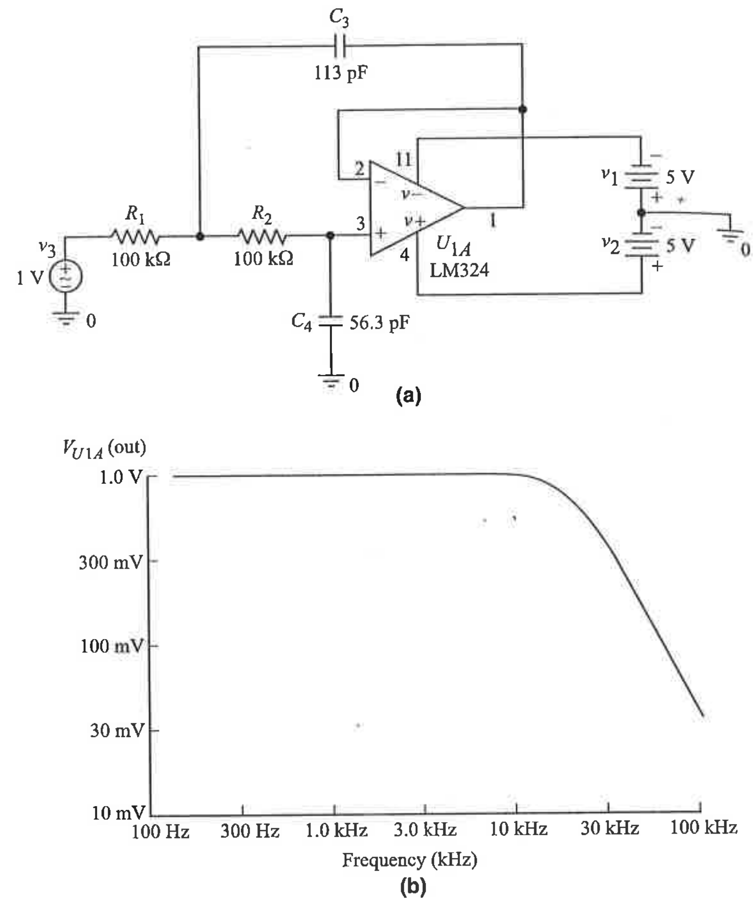

Figure 15.8 (a) Circuit used in the computer simulation of the design in Example 15.1; (b) output versus frequency

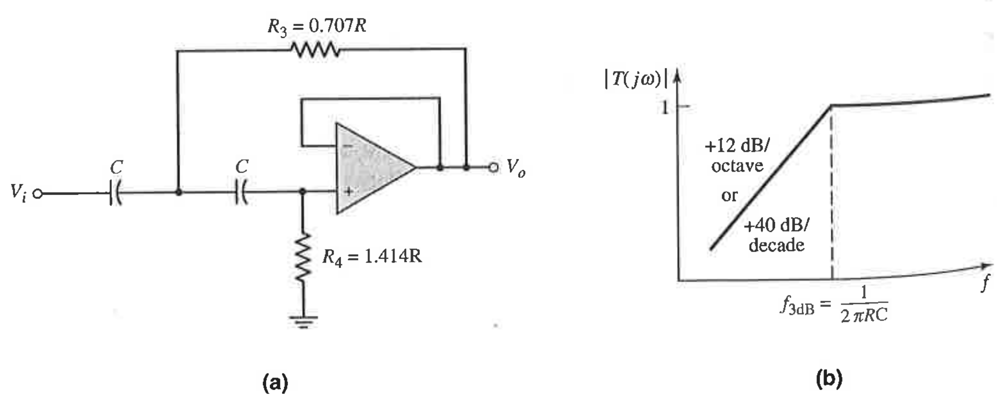

Figure 15.9 (a) Two-pole high-pass Butterworth filter and (b) Bode plot, transfer function magnitude

except that the derivative is set equal to zero at $s = j\omega = \infty$. Also, the two capacitors are set equal to each other. The 3 dB or cutoff frequency can be written in the general form

$$\omega_{3\,\text{dB}} = 2\pi f_{3\,\text{dB}} = \frac{1}{RC} \tag{15.22}$$

We find that $R_3 = 0.707R$ and $R_4 = 1.414R$. The magnitude of the voltage transfer function for the two-pole high-pass Butterworth is

$$|T| = \frac{1}{\sqrt{1 + \left(\dfrac{f_{3\,\text{dB}}}{f}\right)^4}} \tag{15.23}$$

The Bode plot of the transfer function magnitude for the two-pole high-pass Butterworth filter is shown in Figure 15.9(b).

15.1.5 Higher-Order Butterworth Filters

The filter order is the number of poles and is usually dictated by the application requirements. An N-pole active low-pass filter has a high-frequency rolloff rate of $N \times 6$ dB/octave. Similarly, the response of an N-pole high-pass filter increases at a rate of $N \times 6$ dB/octave, up to the cutoff frequency. In each case, the 3 dB frequency is defined as

$$f_{3\,\text{dB}} = \frac{1}{2\pi RC} \tag{15.24}$$

The magnitude of the voltage transfer function for a Butterworth Nth-order low-pass filter is

$$|T| = \frac{1}{\sqrt{1 + \left(\dfrac{f}{f_{3\,\text{dB}}}\right)^{2N}}} \tag{15.25}$$

For a Butterworth Nth-order high-pass filter, the voltage transfer function magnitude is

$$|T| = \frac{1}{\sqrt{1 + \left(\dfrac{f_{3\,\text{dB}}}{f}\right)^{2N}}} \tag{15.26}$$

Figure 15.10(a) shows a three-pole low-pass Butterworth filter. The three resistors are equal, and the relationship between the capacitors is found by taking the first and second derivatives of the voltage gain magnitude with respect to frequency and setting those derivatives equal to zero at $s = j\omega = 0$. Figure 15.10(b) shows a three-pole high-pass Butterworth filter. In this case, the three capacitors are equal and the relationship between the resistors is also found through the derivatives.

Higher-order filters can be created by adding additional RC networks. However, the loading effect on each additional RC circuit becomes more severe. The usefulness of active filters is realized when two or more op-amp filter circuits are cascaded to produce one large higher-order active filter.