Bell's Theorem

1) EPR did not doubt Q.M. was correct. To them it was incomplete.

2) They claimed: "It is an incomplete description of physical reality!"

3) ψ is not the whole story. Some other λ (hidden variables) are needed to characterize the state of a system fully. These hidden variables would help getting rid off the spooky action at a distance

4) In 1964, J. S. Bell proved that any <u>local</u> hidden variable theory is incompatible with Q.M.

Bell's version of the EPR-Bohm experiment

The detectors are oriented in arbitrary directions $\vec{a}$ & $\vec{b}$. These detectors could be Stern-Gerlach interferometers.

The first detector measures the component of the electron spin in unit direction $\vec{a}$. The second measures the spin of the positron in direction $\vec{b}$.

- If we record the spin in units of $\frac{\hbar}{2}$
- Each detector registers the value $+1$ (for spin up) or -1 (spin down), along the direction in question.

A table of results, for many π^0 decays, might look like this.

in units of $\hbar/2$	electron	positron	product
	+1	−1	−1
	+1	+1	+1
	−1	+1	−1
	+1	−1	−1
	−1	−1	+1

This is assuming reality for each spin and no spooky action at a distance!

I'll calculate the average value of the product of the spins, for a given set of detector orientations. We call that average $P(\vec{a},\vec{b})$

If detectors are parallel $\vec{a}=\vec{b}$ This is the original EPRB configuration.

In this case, if one spin is up, the other one is down, or vice-versa, and the product is always −1. So too is the average

$$\boxed{P(\vec{a},\vec{a})=-1}$$

As we will show later, for arbitrary orientations, QM predicts

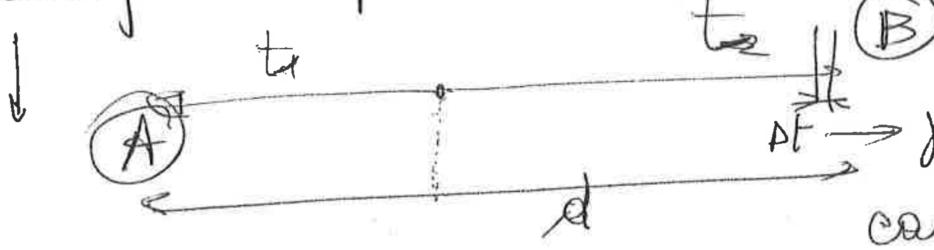
$$P(\vec{a},\vec{b}) = -\vec{a}\cdot\vec{b}$$

Bell proved that this result is impossible in any local hidden variable theory.

<u>Argument</u>

Suppose the complete state of the electron/positron system is characterized by the hidden variables λ. These λ's vary from one pion decay to the next. In other words, there is a distribution of these λ's.

Suppose the outcome of the electron measurement is independent of $\vec{b}$. The latter can be selected at the positron end just before the electron measurement is made, i.e., far too late for any subluminal message to reach the electron detector. This is the locality assumption.



just Δt before set $\vec{b}$. in time Δt, a photon cannot reach detector A even at the speed of light, the fastest way to communicate.

Some function $A(\vec{a},\lambda)$ give the results of the electron measurement. Some $B(\vec{b},\lambda)$ gives the results of the positron measurements.

This is to restore locality in QM introducing hidden variables

$$A(\vec{a},\lambda) = \pm 1$$
$$B(\vec{b},\lambda) = \pm 1$$

The vital assumption is that the result B for positron does not depend on settings of $\vec{a}$, nor A on $\vec{b}$. (locality)

When the detectors are aligned

$$A(\vec{a},\lambda) = -B(\vec{a},\lambda) \quad \forall \lambda \quad \forall \vec{a}$$

hidden variable theory cannot violate conservation of total spin

<u>Average</u>

$$P(\vec{a},\vec{b}) = +\int \rho(\lambda)\, A(\vec{a},\lambda)\, B(\vec{b},\lambda)\, d\lambda$$

$\rho(\lambda)$ = probability density of hidden variable

i.e. $\int \rho(\lambda)\, d\lambda = 1$

$$P(\vec{a},\vec{b}) = -\int \rho(\lambda)\, A(\vec{a},\lambda)\, A(\vec{b},\lambda)\, d\lambda$$

Take any other unit vector $\vec{c}$

$$P(\vec{a},\vec{b}) - P(\vec{a},\vec{c}) = -\int \rho(\lambda)\left[A(\vec{a},\lambda)A(\vec{b},\lambda) - A(\vec{a},\lambda)A(\vec{c},\lambda)\right]d\lambda$$

But $[A(\vec{b},\lambda)]^2 = +1$

$$P(\vec{a},\vec{b}) - P(\vec{a},\vec{c}) = -\int \rho(\lambda)\Big[A(\vec{a},\lambda)\,A(\vec{b},\lambda) - A(\vec{a},\lambda)\,\boxed{A(\vec{b},\lambda)\,A(\vec{b},\lambda)}\,A(\vec{c},\lambda)\Big]d\lambda$$

$$P(\vec{a},\vec{b}) - P(\vec{a},\vec{c}) = -\int \rho(\lambda)\left[1 - A(\vec{b},\lambda)A(\vec{c},\lambda)\right] A(\vec{a},\lambda)A(\vec{b},\lambda)\, d\lambda$$

But

$$-1 \leq A(\vec{a},\lambda)A(\vec{b},\lambda) \leq +1$$

$$[-1 \leq A(\vec{b},\lambda)A(\vec{c},\lambda) \leq +1]$$

~~hence~~ also $\rho(\lambda)\left[1 - A(\vec{b},\lambda)A(\vec{c},\lambda)\right] \geq 0$

$$|P(\vec{a},\vec{b}) - P(\vec{a},\vec{c})| = \left|\int \underbrace{\rho(\lambda)[\quad]}_{\geq 0} \underbrace{A(a,\lambda)A(b,\lambda)}_{\substack{\pm 1 \\ ||\leq 1}} d\lambda\right|$$

$$\leq \int \rho(\lambda)\left[1 - A(\vec{b},\lambda)A(\vec{c},\lambda)\right] d\lambda$$

or

$$|P(\vec{a},\vec{b}) - P(\vec{a},\vec{c})| \leq \underbrace{\int \rho(\lambda)\, d\lambda}_{= 1} - \underbrace{\int \rho(\lambda)\, A(\vec{b},\lambda)A(\vec{c},\lambda)\, d\lambda}$$

$$1 + P(\vec{b},\vec{c})$$

This is Bell's inequality

$\forall \vec{a},\vec{b},\vec{c}$ we must have. $\boxed{|P(\vec{a},\vec{b}) - P(\vec{a},\vec{c})| \leq 1 + P(\vec{b},\vec{c})}$

The quantum-mechanical result

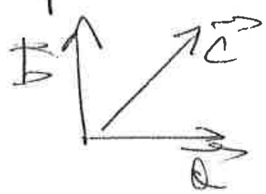
$$P(\vec{a},\vec{b}) = -\vec{a}\cdot\vec{b}$$

is incompatible with this inequality

The inequality above cannot be satisfied $\forall \vec{a},\vec{b},\vec{c}$

Example

Suppose all 3 vectors $\vec{a}, \vec{b}, \vec{c}$ lie in a plane
$\vec{c}$ makes an angle of 45° with $\vec{a}$ & $\vec{b}$.



$$P(\vec{a}, \vec{b}) = 0$$

$$P(\vec{a}, \vec{c}) = P(\vec{b}, \vec{c}) = -0.707$$

This is inconsistent with Bell's inequality.

$$0.707 \not\leq 1 - 0.707 = 0.293$$

If EPR are right, not only is QM incomplete, it is wrong!

On the other hand, if QM is right, then no local hidden variable theory is going to rescue us from the nonlocality Einstein considered so preposterous.

i.e., if there are some hidden variables that QM has failed to include and locality holds then Q.M is wrong.

Quantum Mechanical Calculation of $P(\vec{a},\vec{b})$

$P(\vec{a},\vec{b})$ is the correlation coefficient.

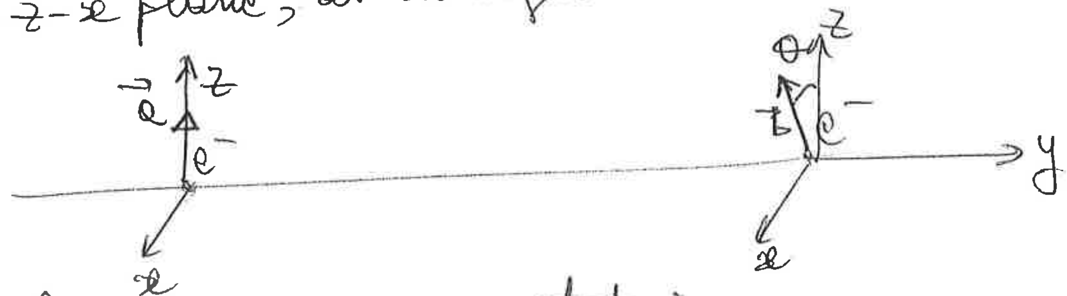
$$\left[\frac{4}{\hbar^2}\vec{S}_{a1} S_{b2}\right]_{av} = \left\langle\left(\frac{4}{\hbar^2}\right) S_{a1} S_{b2}\right\rangle = P(\vec{a},\vec{b})$$

where the average is taken over the singlet state

Assume $\vec{a}$ is along z-direction

$$S_{a1} = \frac{\hbar\vec{\sigma}_1}{2}\cdot\hat{z} = \frac{\hbar\sigma_1^z}{2} \qquad S_{b2} = \frac{\hbar\vec{\sigma}_2}{2}\cdot\vec{b}$$

$\vec{b}$ is in the z-x plane, at an angle θ with z-axis



The spinor for the zero-spin state is

$$\frac{1}{\sqrt{2}}\left(|+\rangle|-\rangle - |-\rangle|+\rangle\right)$$

$$\rightarrow P(\vec{a},\vec{b}) = \frac{1}{2}\left(\langle+|\langle-| - \langle-|\langle+|\right)\left(\frac{4}{\hbar^2}S_{z1}\, S_{b2}\right)\left(|+\rangle|-\rangle - |-\rangle|+\rangle\right)$$

$$= \frac{2}{\hbar^2}\Big(\langle+|S_{z1}|+\rangle\langle-|S_{b2}|-\rangle - \langle+|S_{z1}|-\rangle\langle-|S_{b2}|+\rangle - \langle-|S_{z1}|+\rangle\langle+|S_{b2}|-\rangle + \langle-|S_{z1}|-\rangle\langle+|S_{b2}|+\rangle\Big)$$

Since $\langle+|S_{z1}|-\rangle = \langle-|S_{z1}|+\rangle = 0$ the second & third terms vanish.

But

(8)

$\langle + | S_{z_1} | + \rangle = \frac{\hbar}{2}$ & $\langle - | S_{z_1} | - \rangle = -\frac{\hbar}{2}$

$|+\rangle$ $|-\rangle$ are not eigenstates of $S_{b_2} = \frac{\hbar}{2} \vec{\sigma}_2 \cdot \vec{b}$

Eigenvectors of $\vec{\sigma}_2 \cdot \vec{b}$

are $|\xi_b^+\rangle = \cos\frac{\theta}{2} |+\rangle + e^{i\varphi} \sin\left(\frac{\theta}{2}\right) |-\rangle$ $\quad \varphi = 0.$

$|\xi_b^+\rangle = \cos\frac{\theta}{2} |+\rangle + \sin\frac{\theta}{2} |-\rangle$; with eigenvalue $+1$

$|\xi_b^-\rangle = -\sin\frac{\theta}{2} |+\rangle + \cos\frac{\theta}{2} |-\rangle$; " " -1

~~$\langle + | S_{b_2} | + \rangle = \frac{\hbar}{2}$~~

$S_{b_2} = \frac{\hbar}{2} (\vec{\sigma}_2 \cdot \vec{b}) = \frac{\hbar}{2} \left(\sigma_{2x} \sin\theta + \sigma_{2z} \cos\theta \right)$

$\langle + | S_{b_2} | + \rangle = \frac{\hbar}{2} \left[\sin\theta \overbrace{\langle + | \sigma_{2x} | + \rangle}^{=0} + \cos\theta \underbrace{\langle + | \sigma_{2z} | + \rangle}_{+1} \right]$

$(1\ 0)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$
$= (1\ 0)\begin{pmatrix} 0 \\ 1 \end{pmatrix}$
$= 0$

$\langle + | S_{b_2} | + \rangle = \frac{\hbar}{2} \cos\theta$

$\langle - | S_{b_2} | - \rangle = \frac{\hbar}{2} \left[\sin\theta \overbrace{\langle - | \sigma_{2x} | - \rangle}^{=0} + \cos\theta \underbrace{\langle - | \sigma_{2z} | - \rangle}_{=-1} \right]$

$\langle - | S_{b_2} | - \rangle = -\frac{\hbar}{2} \cos\theta$

$\Rightarrow P(\vec{a}, \vec{b}) = \frac{2}{\hbar^2} \left[\frac{\hbar}{2} \left(-\frac{\hbar}{2} \cos\theta \right) - \frac{\hbar}{2} \frac{\hbar}{2} \cos\theta \right]$

$$\boxed{P(\vec{a}, \vec{b}) = -\cos\theta}$$

There are values of $\vec{a}$ & $\vec{b}$ for which $P(\vec{a},\vec{b})$ calculated quantum-mechanically can violate ~~do not satisfy~~ Bell's inequality.

Take $\vec{a}$ along z

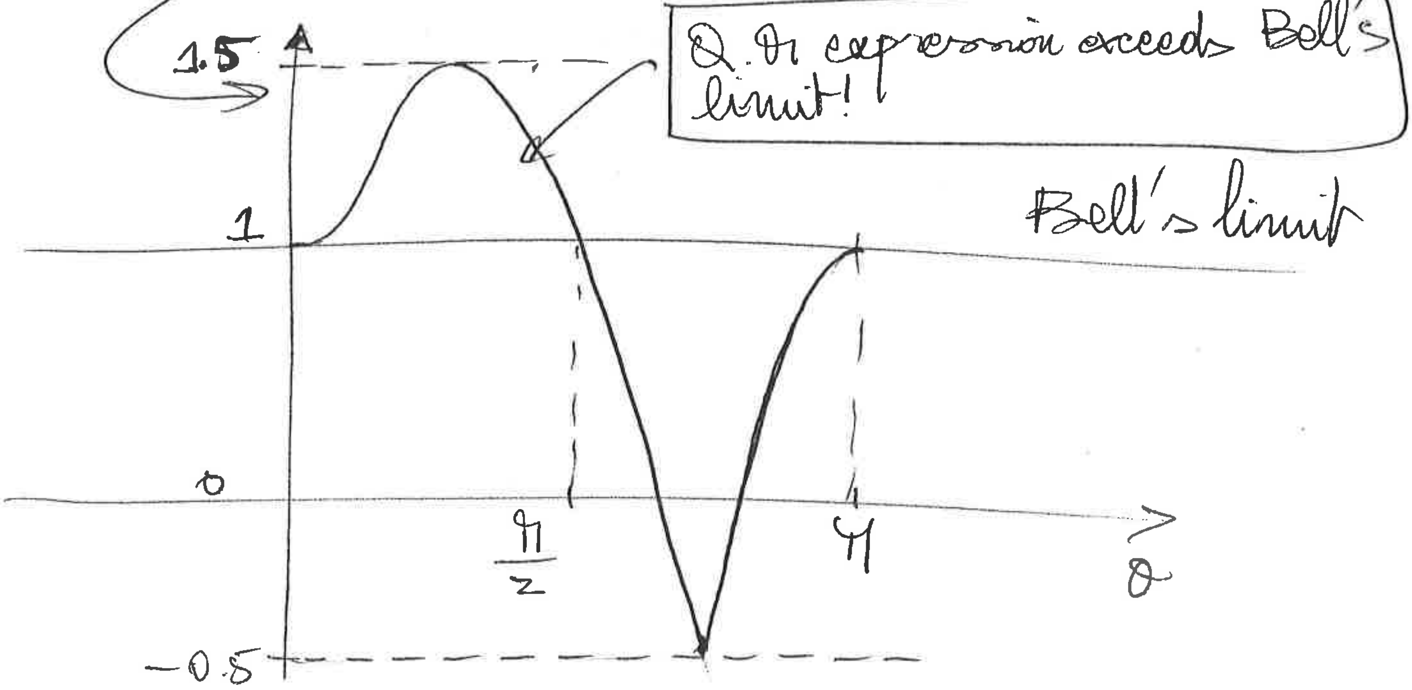
$\vec{b}$ in (x,z) plane at an angle θ with z

$\vec{c}$ in " " 2θ with z

$$P(\vec{a},\vec{b}) = -\cos\theta$$
$$P(\vec{a},\vec{c}) = -\cos 2\theta$$
$$P(\vec{b},\vec{c}) = -\cos\theta$$

The quantum-mechanical expression for the left side of the inequality is

$$|P(\vec{a},\vec{b}) - P(\vec{a},\vec{c})| - P(\vec{b},\vec{c})$$
$$= |-\cos\theta + \cos 2\theta| + \cos\theta$$



Bell's inequality provides us with a way to discriminate experimentally between the predictions of QM and those of local hidden-variable theories.

Before Bell's theorem, such a discrimination was thought to be nearly impossible, since hidden-variable theories are designed to mimic the results of QM as best as they can.

Nonlocality

Its converse, locality, is the principle that an event which happens at one place can't instantaneously affect an event at some other place.

EPR QM predicts a breakdown of locality

Bell: results predicted by QM can not be explained by any theory which preserves locality. In other words if the results of an EPR expt agree with QM, there is no way locality can be true. Experiments did indeed agree with predictions of QM. In short, locality is dead.

Bell's theorem IS NOT incompatible with relativity's prediction that nothing can travel at the speed of light.