From Schaum's Outline Series
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## Solved Problems

### BLACKBODY RADIATION

The walls of a cavity maintained at a temperature $T$ continuously emit and absorb electromagnetic radiation (photons), and in equilibrium the amounts of energy emitted and absorbed by the walls are equal. The radiation inside the cavity can be analyzed by opening a small hole in one of the walls of the cavity; the escaping photons constitute what is called *blackbody radiation*. Quantum physics was born when Max Planck discovered around 1900 the correct expression for the experimentally observed spectral distribution of blackbody radiation, i.e. the fraction of the total radiated energy with frequency between $\nu$ and $\nu + d\nu$.

**38.1.** Consider, for simplicity, a cubic cavity of side $l$ whose edges define a set of axes and whose walls are maintained at a temperature $T$. Maxwell's equations for electromagnetic waves show that the rectangular components of the wave vector

$$\mathbf{k} = \frac{2\pi}{\lambda^2}\boldsymbol{\lambda}$$

must satisfy the boundary conditions

$$\frac{k_x l}{\pi} = n_x \qquad \frac{k_y l}{\pi} = n_y \qquad \frac{k_z l}{\pi} = n_z$$

where $n_x$, $n_y$, and $n_z$ are positive integers. (These boundary conditions ensure an integral number of half waves in each edge of the cube.) Each triplet $(n_x, n_y, n_z)$ represents, in classical terms, an electromagnetic *mode of oscillation* for the cavity; we shall consider these modes as photon *states*. Find the number of modes in the frequency interval between $\nu$ and $\nu + d\nu$, given that there are two independent polarization directions for each mode.

Each allowed frequency $\nu$ corresponds to a certain mode $(n_x, n_y, n_z)$ and can be written as

$$\nu = \frac{c}{\lambda} = \frac{ck}{2\pi} = \frac{c}{2l}\sqrt{n_x^2 + n_y^2 + n_z^2} = \frac{cN}{2l}$$

where $N = \sqrt{n_x^2 + n_y^2 + n_z^2}$. Instead of trying to find the number of integer states $(n_x, n_y, n_z)$ corresponding to a given $N$, we use a continuous approximation and find the number of points in a spherical shell of radius $N$ and thickness $dN$ in the first octant of $(n_x, n_y, n_z)$-space. This is simply the "volume"

$$dM = \frac{1}{8}(4\pi N^2\, dN) = \frac{\pi}{2}N^2\, dN$$

But the number of states between $N$ and $N + dN$ must equal the number of states in the corresponding frequency interval, $\nu$ to $\nu + d\nu$. Using $\nu = cN/2l$,

$$dN = \frac{2l}{c}\, d\nu$$

from which

$$dM = \frac{\pi}{2}\left(\frac{2l\nu}{c}\right)^2\left(\frac{2l}{c}\, d\nu\right) = \frac{4\pi l^3}{c^3}\nu^2\, d\nu = \frac{4\pi V}{c^3}\nu^2\, d\nu$$

where $V$ is the volume of the cavity. Because there are two possible independent polarization directions for each mode, we must multiply $dM$ by two to obtain the number of photon states $dS$:

$$dS = 2\, dM = \frac{8\pi V}{c^3}\nu^2\, d\nu = g(\nu)\, d\nu$$

Thus the density of photon states in a frequency interval $d\nu$ is

$$g(\nu) = \frac{8\pi V}{c^3}\nu^2$$

**38.2.** Viewing the cavity of Problem 38.1 as a container of photons, which are spin-1 bosons, determine the spectral distribution (the amount of energy per frequency interval) of the blackbody radiation coming from a small hole in the container. Because photons are continuously being emitted and absorbed by the walls of the cavity, their number is not conserved.

Using the density of states $g(\nu)$ determined in Problem 38.1 and the Bose-Einstein distribution function for photons, (*38.2*) with $\alpha = 0$, we find the number of photons, $dn_\nu$, with frequencies between $\nu$ and $\nu + d\nu$ to be

$$dn_\nu = F_{BE}\, g(\nu)\, d\nu = \frac{1}{e^{E/kT} - 1} \frac{8\pi V}{c^3} \nu^2\, d\nu$$

Each photon has an energy $E = h\nu$, so the amount of energy, $dE_\nu$, carried by the $dn_\nu$ photons is

$$dE_\nu = h\nu\, dn_\nu = \frac{8\pi V h}{c^3} \frac{\nu^3}{e^{h\nu/kT} - 1}\, d\nu = F(\nu)\, d\nu$$

The spectral distribution, $F(\nu)$, is plotted in Fig. 38-1. It should be mentioned that Planck arrived at the function $F(\nu)$ by a different route.

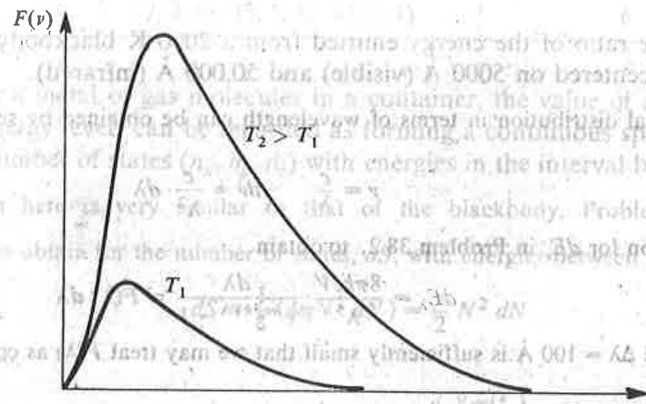


**Fig. 38-1**

**38.3.** The *Stefan-Boltzmann law* states that the total electromagnetic energy inside a cavity whose walls are maintained at a temperature $T$ is proportional to $T^4$. Show how the law follows from the result of Problem 38.2, and evaluate the proportionality factor.

The total energy in the cavity of Problem 38.2 is given by

$$E = \int dE_\nu = \frac{8\pi V h}{c^3} \int_0^\infty \frac{\nu^3\, d\nu}{e^{h\nu/kT} - 1} = \frac{8\pi V k^4 T^4}{c^3 h^3} \int_0^\infty \frac{q^3\, dq}{e^q - 1} = \text{constant} \times T^4$$

From (*38.5*), the integral has the value

$$\Gamma(4)\zeta(4) = 3!\, \frac{\pi^4}{90} = 6.49$$

and so

$$E = \frac{8\pi k^4 (6.49)}{(hc)^3} V T^4 = \frac{8\pi (8.617 \times 10^{-5}\ \text{eV/K})^4 (6.49)}{(12.4 \times 10^{-7}\ \text{eV} \cdot \text{m})^3} V T^4$$

$$= (4.71\ \text{keV/K}^4 \cdot \text{m}^3) V T^4$$

**38.4.** From Fig. 38-1 it is seen that the peak of the spectral distribution curve shifts upward in frequency as the temperature increases. The *Wien displacement law* states that

$$\lambda_{max} T = \text{constant}$$

where $\lambda_{max}$ is the wavelength at which the maximum value of $F(\nu)$ occurs. Derive the Wien displacement law.

Setting

$$\frac{dF(\nu)}{d\nu} = \frac{d}{d\nu}\left[\frac{8\pi Vh}{c^3}\frac{\nu^3}{e^{h\nu/kT}-1}\right] = \frac{8\pi Vh}{c^3}\frac{\nu^2\left[e^{h\nu/kT}\left(3-\frac{h\nu}{kT}\right)-3\right]}{(e^{h\nu/kT}-1)^2} = 0$$

we obtain, for a maximum,

$$e^{h\nu_{max}/kT}\left(3-\frac{h\nu_{max}}{kT}\right)-3 \equiv e^y(3-y)-3=0$$

This transcendental equation for $y = h\nu_{max}/kT$, which must be solved by approximation methods, will have some solution

$$y = \frac{h\nu_{max}}{kT} = \text{constant}$$

or, since $\nu_{max} = c/\lambda_{max}$,

$$\lambda_{max}T = \text{constant}$$

**38.5.** Determine the ratio of the energy emitted from a 2000 K blackbody in wavelength bands of width 100 Å centered on 5000 Å (visible) and 50,000 Å (infrared).

The spectral distribution in terms of wavelength can be obtained by setting (working with magnitudes only)

$$\nu = \frac{c}{\lambda} \qquad d\nu = \frac{c}{\lambda^2}\,d\lambda$$

in the expression for $dE_\nu$ in Problem 38.2, to obtain

$$dE_\lambda = \frac{8\pi hcV}{\lambda^5}\frac{d\lambda}{e^{hc/kT\lambda}-1} = F(\lambda)\,d\lambda$$

The bandwidth $\Delta\lambda = 100$ Å is sufficiently small that we may treat $F(\lambda)$ as constant over this interval, to obtain

$$\frac{\Delta E_{50,000}}{\Delta E_{5000}} = \frac{(5000\ \text{Å})^5}{(50{,}000\ \text{Å})^5}\,\frac{e^{(12{,}400\ \text{eV}\cdot\text{Å})/(8.62\times10^{-5}\ \text{eV/K})(2000\ \text{K})(5000\ \text{Å})}-1}{e^{(12{,}400\ \text{eV}\cdot\text{Å})/(8.62\times10^{-5}\ \text{eV/K})(2000\ \text{K})(50{,}000\ \text{Å})}-1} = 5.50$$

This result shows that only a small amount of the overall energy is radiated as visible light.