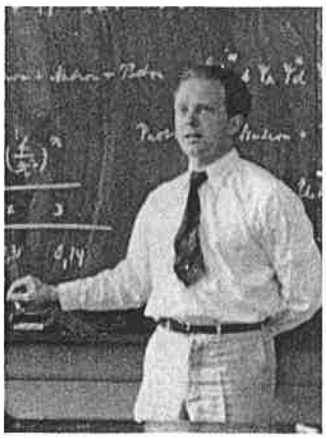
From J.R. Taylor - C.D. Zafiratos & M.A. Dubson
Modern Physics
For Scientists & Engineers - 2nd Edition

6.8 The Uncertainty Relation for Position and Momentum

Werner Heisenberg
(1901–1976, German)



After earning his PhD at Munich, Heisenberg worked with Born and then Bohr. His many contributions to modern physics include an early formulation of quantum mechanics in terms of matrices, several ideas in nuclear physics, and the famous uncertainty principle, for which he won the 1932 Nobel Prize in physics. He remained in Germany during World War II and worked on nuclear reactor design. The possibility that he might be working on an atomic bomb for the Nazis so frightened the Allies that a plan was devised to have him assassinated. An American agent, named Moe Berg, posed as a physicist and met with Heisenberg when he was visiting neutral Switzerland in late 1944. After talking with Heisenberg, agent Berg decided that the Germans had made little progress toward a bomb and chose not to kill him.

We have seen that the wave function of a single particle is spread out over some interval. This means that a measurement of the particle's position x may yield any value within this interval. (To simplify our discussion, we suppose that our particle moves in one dimension and so has a single coordinate x.) Therefore, the particle's position is *uncertain* by an amount $\pm\Delta x$, and we refer to Δx as the *uncertainty in the position*. As mentioned at the end of Section 6.4, the standard interpretation of quantum mechanics is that this uncertainty is not just a reflection of our ignorance of the particle's position. Rather, the particle *does not have a definite position*. The uncertainty exists in nature, not just in the mind of the physicist. The uncertainty Δx can be smaller in some states than in others; but for any given state, specified by a wave function $\Psi(x,t)$, there is some nonzero interval within which the particle may be found, and the particle's position is simply not defined any more precisely than that.

In Section 6.7 we saw that the wave function that describes a particle can be built up from sinusoidal waves, but that this requires a spread of different wave numbers k (or wavelengths λ). From the de Broglie relation,

$$p = \frac{h}{\lambda} = \hbar k$$

it follows that a spread of wave numbers implies a spread of momenta; that is, the particle's momentum p, like its position x, is uncertain. A measurement of the momentum may yield any of several values in a range given by

$$\Delta p = \hbar\,\Delta k \qquad \textbf{(6.33)}$$

We have seen that the spreads Δx and Δk are not independent, but always satisfy the inequality (6.31),

$$\Delta x\,\Delta k \ge \tfrac{1}{2}$$

If we multiply this relation by $\hbar$, we find that

$$\Delta x\,\Delta p \ge \frac{\hbar}{2} \qquad \textbf{(6.34)}$$

This is one of several inequalities called the **Heisenberg uncertainty relations** and known collectively as the **Heisenberg uncertainty principle**. It implies that both the position and momentum of a particle have uncertainties in the sense just described. One can find states for which Δx is small, but (6.34) tells us that Δp will be large; one can also find states for which Δp is small, but Δx will be large. In all cases their product, $\Delta x\,\Delta p$, will never be less than $\hbar/2$.

In classical physics it was taken for granted that particles have definite values of their position x and momentum p. It was recognized, of course, that x and p could not be measured with perfect accuracy. But it was assumed that with enough care, one could make both experimental uncertainties as small as one pleased. Heisenberg's uncertainty relation (6.34) shows that these assumptions were incorrect. There are intrinsic uncertainties, or spreads, Δx and Δp in the position and momentum of any particle. Whereas either one of Δx and Δp can be made as small as one pleases, their product can never be less than $\hbar/2$.

We now know that the uncertainty principle applies to all particles. On the macroscopic level, however, it is seldom important, as the following example illustrates.

Example 6.3

The position x of a 0.01-g pellet has been carefully measured and is known within $\pm 0.5\ \mu\text{m}$. According to the uncertainty principle, what are the minimum uncertainties in its momentum and velocity, consistent with our knowledge of x?

If x is known within $\pm 0.5\ \mu\text{m}$, the spread $\pm \Delta x$ in the position is certainly no larger than $0.5\ \mu\text{m}$:

$$\Delta x \le 0.5\ \mu\text{m}$$

According to the uncertainty relation (6.34), this implies that the momentum is uncertain by an amount

$$\Delta p \ge \frac{\hbar}{2\,\Delta x} \ge \frac{10^{-34}\ \text{J}\cdot\text{s}}{10^{-6}\ \text{m}} = 10^{-28}\ \text{kg}\cdot\text{m/s}$$

Therefore, the velocity $v = p/m$ is uncertain by*

$$\Delta v = \frac{\Delta p}{m} \ge \frac{10^{-28}\ \text{kg}\cdot\text{m/s}}{10^{-5}\ \text{kg}} = 10^{-23}\ \text{m/s}$$

Clearly, the inevitable uncertainties in p and v required by the uncertainty principle are of no practical importance in this case. (To appreciate how small 10^{-23} m/s is, notice that at this speed our pellet would take about a million years to cross an atomic diameter.)

Although the uncertainty principle is seldom important on the macroscopic level, it is frequently very important on the microscopic level, as the next example illustrates.

*The mass of a stable particle has no uncertainty, so we can treat m as a constant in the relation $v = p/m$.

Example 6.4

An electron is known to be somewhere in an interval of total width $a \approx 0.1$ nm (the size of a small atom). What is the minimum uncertainty in its velocity, consistent with this knowledge?

If we know the electron is certainly inside an interval of total width a,

$$\Delta x \leq \frac{a}{2} \tag{6.35}$$

(Remember that Δx is the spread from the central value out to either side.) According to the uncertainty relation (6.34), this implies that

$$\Delta p \geq \frac{\hbar}{2\,\Delta x} \geq \frac{\hbar}{a} \tag{6.36}$$

This implies that $\Delta v = \Delta p/m \gtrsim \hbar/(am)$ or

$$\Delta v \geq \frac{\hbar c^2}{amc^2} = \frac{200\text{ eV}\cdot\text{nm}}{(0.1\text{ nm})\times(0.5\times10^6\text{ eV})}c = \frac{c}{250} \approx 10^6\text{ m/s}$$

(where we multiplied numerator and denominator by c^2 to take advantage of the useful combinations $\hbar c$ and mc^2). This large uncertainty in v shows the great importance of the uncertainty principle for systems with atomic dimensions.

Perhaps the most dramatic consequence of the uncertainty principle is that a particle confined in a small region cannot be exactly at rest, since if it were, its momentum would be precisely zero, which would violate (6.36). Since its momentum cannot be precisely zero, the same is true of its kinetic energy. Therefore, the particle has a minimum kinetic energy, which we can estimate as follows: Since the momentum is spread out by an amount given by (6.36) as

$$\Delta p \geq \frac{\hbar}{a} \tag{6.37}$$

the magnitude of p must be, on average, at least of this same order. Thus the kinetic energy, whether it has a definite value or not, must on average have magnitude

$$\langle K\rangle = \left\langle\frac{p^2}{2m}\right\rangle \gtrsim \frac{(\Delta p)^2}{2m} \tag{6.38}$$

or, by (6.37)

$$\langle K\rangle \gtrsim \frac{\hbar^2}{2ma^2} \tag{6.39}$$

The energy (6.39) is called the **zero-point energy**. It is the minimum possible kinetic energy for a quantum particle confined inside a region of width a. The kinetic energy can, of course, be larger than this, but it cannot be any smaller.

Example 6.5

What is the minimum kinetic energy of an electron confined in a region of width $a \approx 0.1$ nm, the size of a small atom?

According to (6.39),

$$\langle K \rangle \gtrsim \frac{\hbar^2}{2ma^2} = \frac{(\hbar c)^2}{(2mc^2)a^2} = \frac{(200\,\mathrm{eV\cdot nm})^2}{(10^6\,\mathrm{eV}) \times (0.1\,\mathrm{nm})^2} = 4\,\mathrm{eV}$$

This lower bound is satisfactorily consistent with the known kinetic energy, 13.6 eV, of an electron in the ground state of a hydrogen atom.*

The bound (6.39) gives a useful estimate of the minimum kinetic energy of several other systems. For more examples, see Problems 6.39, 6.42, and 6.43.

We have so far written the uncertainty relation only for the case of a particle moving in one dimension. In three dimensions there is a corresponding inequality for each dimension separately:

$$\Delta x\, \Delta p_x \ge \frac{\hbar}{2}, \qquad \Delta y\, \Delta p_y \ge \frac{\hbar}{2}, \qquad \Delta z\, \Delta p_z \ge \frac{\hbar}{2} \tag{6.40}$$

where x, y, z are the particle's three coordinates and p_x, p_y, p_z, the three components of its momentum. (See Problem 6.42 for an application.)

Heisenberg's Microscope

The uncertainty principle can be illustrated by several thought experiments, the best known of which is sometimes called the Heisenberg microscope. In this thought experiment a classical physicist — reluctant to accept the uncertainty principle — tries to disprove it by showing that he can measure the position and momentum of a particle with uncertainties smaller than are allowed by the uncertainty relation (6.34).

To find the position x of the particle, our classical physicist observes it with a microscope, as shown in Fig. 6.16. Now, it is a fact — well known in classical physics — that the resolution of any microscope is limited by the diffraction of light. Specifically, the angular resolution $\theta_{\min}$ (the minimum angle at which two points can be told apart) is given by the so-called Rayleigh criterion,

$$\theta_{\min} \approx \frac{\lambda}{d} \tag{6.41}$$

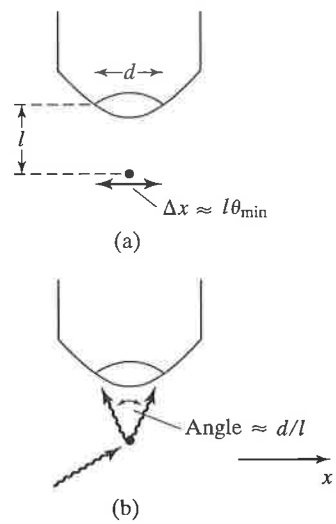


FIGURE 6.16
The Heisenberg microscope. **(a)** The minimum experimental uncertainty in the particle's position x is determined by the microscope's resolution as $\Delta x \approx l\,\theta_{\min} \approx l\lambda/d$. **(b)** The direction of a photon entering the microscope is uncertain by an angle of order d/l; therefore, the photon gives the particle a momentum (in the x direction) which is uncertain by $\Delta p_x \approx p_\gamma d/l$.

*Recall that we saw in Sec. 5.6 that the kinetic energy is the negative of the total energy ($E = -13.6$ eV). Also, here we have treated an electron in one dimension. If one uses the inequalities (6.40) to include the motion in all three dimensions, one finds $\langle K \rangle \gtrsim 12$ eV, in excellent agreement with the observed 13.6 eV.

where λ is the wavelength of light used and d the diameter of the objective lens. If the particle is a distance l below the lens, the minimum uncertainty in x is [see Fig. 6.16(a)]

$$\Delta x \approx l\theta_{\min} \approx \frac{l\lambda}{d} \tag{6.42}$$

(where we assume for simplicity that all angles are small, so that $\sin\theta \approx \theta$). Our classical physicist is aware of this limitation, but points out that he can make Δx as small as he pleases, for example, by using light of very short wavelength λ.

Simply to pin down the particle's position with arbitrarily small Δx does not itself conflict with the uncertainty principle. Our classical physicist must show that he can also know the momentum with a suitably small uncertainty; and if we recall that light is quantized, we can quickly show that this is impossible: In order to observe the particle, he must allow at least one photon to strike it, and this collision will change the particle's momentum. He has no way of knowing which part of the lens the photon passed through since the lens sends *any* light from the object through the same image point. Therefore, the direction in which the photon approached the lens is uncertain by an angle of order d/l. [See Fig. 6.16(b).] This means that the x component of the photon's momentum is uncertain by an amount of order $p_\gamma d/l$. Since the particle was struck by the photon, the x component of the particle's momentum is now uncertain by at least this same amount; that is,

$$\Delta p_x \gtrsim p_\gamma \frac{d}{l} = \frac{h}{\lambda}\cdot\frac{d}{l} \tag{6.43}$$

Our classical physicist can make this uncertainty in p_x as small as he pleases, for example by making λ large. But comparing (6.42) and (6.43), we see that whatever he does to reduce Δp_x will increase Δx and vice versa. In particular, multiplying (6.42) by (6.43), we find that

$$\Delta x\,\Delta p_x \gtrsim h \tag{6.44}$$

and our classical physicist has failed in his attempt to disprove the uncertainty principle.*

The uncertainty principle is a general result that follows from the particle-wave duality of nature. We should emphasize that our analysis of the Heisenberg microscope is not an alternative proof of this general result; it serves only to illustrate the inevitable appearance of the uncertainty principle in the context of one particular experiment.

6.9 The Uncertainty Relation for Time and Energy

Just as the inequality $\Delta x\,\Delta k \geq \frac{1}{2}$ implies the position-momentum uncertainty relation, $\Delta x\,\Delta p \geq \hbar/2$, so the inequality (6.32)

$$\Delta t\,\Delta\omega \geq \frac{1}{2} \tag{6.45}$$

*The fact that we have found $\Delta x\,\Delta p_x \gtrsim h$, rather than $\hbar/2$, is not significant, since the arguments leading to (6.44) were only order-of-magnitude arguments.

implies a corresponding relation for time and energy. Specifically, if we multiply by $\hbar$, we find the **time-energy uncertainty relation**

$$\Delta t \, \Delta E \geq \frac{\hbar}{2} \tag{6.46}$$

Here ΔE is the uncertainty in the particle's energy: A quantum particle generally does not have a definite energy, and measurement of its energy can yield any answer within a range $\pm\Delta E$. To understand the significance of Δt, recall that the inequality (6.45) arose when we considered a wave pulse as a function of time t at one fixed position x. The time Δt characterizes the time spent by the pulse at that position. Thus, for a quantum wave, Δt characterizes the time for which the particle is likely to be found at the position x. According to (6.46), if Δt is small, the particle must have a large uncertainty ΔE in its energy and vice versa.

If a particle has a definite energy, then $\Delta E = 0$, and (6.46) tells us that Δt must be infinite. That is, a quantum particle with definite energy stays localized in the same region (and in the same state, in fact) *for all time.* States with this property are the quantum analog of Bohr's stationary orbits and are called stationary states, as we discuss in Chapter 7.

If a particle (or, more generally, any quantum system) does *not* remain in the same state forever, Δt is finite and (6.46) tells us that ΔE cannot be zero; that is, the energy must be uncertain. For example, any unstable state of an atom or nucleus lives for a certain finite time Δt, after which it decays by emitting a particle (an electron, photon, or α particle, for example). This means that the energy of any unstable atom or nucleus has a minimum uncertainty*

$$\Delta E \approx \frac{\hbar}{2\,\Delta t} \tag{6.47}$$

Since the energy of the original unstable state is uncertain, the same is true of the ejected particle. In some cases one can measure both the spread of energies of the ejected particles (from many decays of identical unstable systems) and the lifetime Δt; one can then confirm the relation (6.47). In many applications one measures one of the quantities ΔE or Δt, then uses (6.47) to estimate the other.

Example 6.6

Many excited states of atoms are unstable and decay by emission of a photon in a time of order $\Delta t \approx 10^{-8}$ s. What is the minimum uncertainty in the energy of such an atomic state?

According to (6.47), the minimum uncertainty in energy is

$$\Delta E \approx \frac{\hbar}{2\Delta t} = \frac{\hbar c}{2c\Delta t} \approx \frac{200 \text{ eV} \cdot \text{nm}}{2 \times (3 \times 10^{17} \text{ nm/s}) \times (10^{-8} \text{ s})} \approx 3 \times 10^{-8} \text{ eV}$$

*In (6.47) we have used the symbol $\approx$ because several different definitions of ΔE and Δt are commonly used. For example, Δt can be defined as the half-life (discussed in Section 1.9) or the mean life (to be discussed in Chapter 17), and the precise form of the relation depends on which definition we adopt. For the case of an unstable particle, (6.47) is exact if we take ΔE to be the so-called half-width at half-height and Δt to be the mean life.

Compared to the several eV between typical atomic energy levels, this uncertainty ΔE is very small. Nevertheless, the resulting spread in the energy, and hence frequency, of the ejected photon is easily measurable with a modern spectrometer.

Nowadays, the frequencies of photons ejected in atomic transitions are used as standards for the definition and calibration of frequency and time. Because of the uncertainty principle, the frequency of any such photon is uncertain by an amount

$$\Delta\omega = \frac{\Delta E}{\hbar} \approx \frac{1}{2\,\Delta t}$$

where Δt is the lifetime of the emitting state. Therefore, it is important to choose atomic states with very long lifetimes Δt to use as standards.

1.4 *Particle vs. Wave; the Uncertainty Relations*

From H. C. Ohanian Principles of Quantum Mechanics (Prentice Hall).

In classical mechanics, a particle is a pointlike mass. At each instant of time, such a classical particle has a well-defined position **r**(t). The motion of the particle proceeds along a well-defined trajectory, and the motion is completely described by specifying how the position **r**(t) varies with time. In contrast, a classical wave is an extensive disturbance in a medium. The medium may consist of a distribution of particles (for example, air serves as the medium for sound waves) or it may consist of classical fields (for example, the electric and magnetic fields surrounding a charge serve as the medium for electromagnetic waves; the wave may be regarded as a propagating kink in the field lines surrounding the charge). Such a disturbance in a medium is endowed with energy and with momentum, and the motion of the disturbance transports this energy and this momentum through the medium. However, the wave has no well-defined position and no well-defined trajectory. Only under exceptional circumstances, when the wavelength is very short compared with the relevant dimensions of any obstacles or apertures, is it possible to identify an approximate trajectory for the wave (in geometrical optics, such an approximate trajectory is called a ray). But even when the wave has an approximate trajectory, the wave can still be distinguished from a particle by its characteristic interference and diffraction effects—when several waves come together, they combine constructively or destructively according to their phase relationships, and when a wave passes through a small aperture, it deflects and spreads out into the shadow zone.

Nineteenth-century physicists observed interference and diffraction effects in light, and they found that electrons appeared to follow definite trajectories in cathode-ray tubes. They therefore concluded that light is a wave and that electrons are particles. The discovery of particle properties of light and the discovery of interference and diffraction effects of electrons demolished this tidy distinction between particles and waves. Light sometimes behaves like a classical particle, and sometimes like a classical wave. Electrons—as well as protons, neutrons, and other "particles"—sometimes behave like classical particles, and sometimes like classical waves. Whether electrons exhibit particle or wave properties depends on circumstances—it depends on what measurement we perform. For instance, in Thomson's electron-diffraction experiment, the electrons behave like waves while passing through the crystallites in the metallic film; but they behave like particles when they strike the fluorescent screen. The pattern of rings shown in Fig. 1.2 is made up of the impacts of many electrons on the fluorescent screen; each individual electron impact yields a pointlike flash of light, as expected for the impact of a particle. Electrons are said to exhibit *duality*: they have the properties of both classical particles and classical waves. However, as Heisenberg has emphasized, electrons are entities of one kind, and the apparent duality arises from the limitations of our language and our intuition. Our language was invented to describe the processes we observe in our everyday life, and all such processes involve macroscopic bodies with a large number of atoms. The processes that occur at the atomic level are outside the realm of our direct experience. We lack the words to describe these processes, and we lack the intuition to visualize them. If we attempt to describe the behavior of electrons in terms of the familiar concepts of classical particles and classical waves, we find that neither is adequate by itself, and only some particle–wave hybrid will give a crude approximation to the behavior of electrons. In view of this, Eddington proposed that electrons be called *wavicles*; but this quite apt neologism has not gained wide acceptance. In modern physics, electrons (or protons, or neutrons, etc.) are often called quantum-mechanical particles.

Heisenberg recognized that, since a quantum-mechanical particle has wave properties, it cannot have sharply defined position and velocity, or position and momentum. The wave packet describing the state of the quantum-mechanical particle has some width, that is, it spans a spread of positions. Furthermore, since a wave packet is a superposition of a number of harmonic waves, it contains a spread of wavelengths, or a spread of momenta. Heisenberg demonstrated that the spread of positions and the spread of momenta are subject to the inequality

$$\Delta x \, \Delta p_x \geq \tfrac{1}{2}\hbar \qquad (16)$$

The spreads of position and momentum within the wave packet represent uncertainties in the possible outcomes of simultaneous measurements of position or momentum. Thus, Eq. (16) is called *Heisenberg's uncertainty relation.* Similar uncertainty relations are, of course, valid for the y and z components of the position and the momentum. Note that according to Eq. (16), if the position is defined very accurately (small Δx), then the momentum is poorly defined (large Δp_x), and conversely.

Heisenberg illustrated the uncertainty relation with several *Gedankenexperimente.* The simplest of these involves the measurement of the position of an electron by means of a slit. Suppose that a beam of electrons approaches a horizontal slit of vertical width a (see Fig. 1.3). If an electron passes through this slit, then

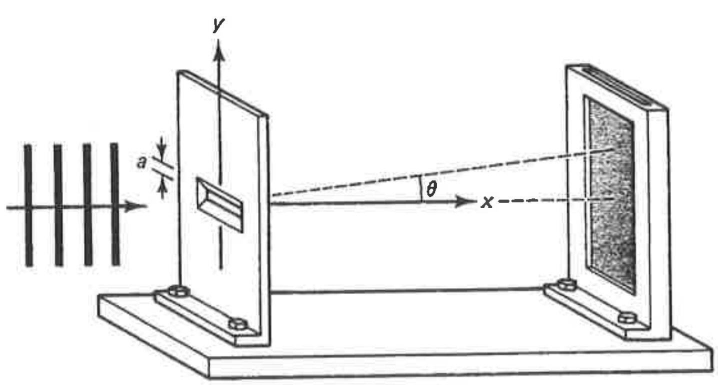


Fig. 1.3 Determination of the vertical position of an electron by means of a horizontal slit.

its vertical position is known to within an uncertainty $\Delta y = a$. However, because of its wave properties, the electron will suffer diffraction at the slit, that is, the electron wave will spread up and down over a range of angles. As a rough measure of the uncertainty of the direction of motion we can take the angular width θ of the central maximum of the diffraction pattern; this angular width is given by an equation familiar from wave optics:

$$a \sin \theta = \lambda \tag{17}$$

The vertical component of the momentum is then uncertain by

$$\Delta p_y = p \sin \theta = \frac{h}{\lambda} \sin \theta \simeq \frac{h}{a} \tag{18}$$

and the product of the simultaneous uncertainties Δy and Δp_y for the measurement of position and momentum in this experimental arrangement is

$$\Delta y \, \Delta p_y = a \frac{h}{a} = h \tag{19}$$

This is consistent with the uncertainty relation (16). Note that Eq. (18) shows quite explicitly how the uncertainty in momentum is affected by the choice of a. If we make a small, and thereby achieve a precise measurement of position, then Δp_y will become large.

Another *Gedankenexperiment* by Heisenberg seeks to measure the position of an electron by means of a hypothetical microscope operating with light of extremely short wavelength, or gamma rays. Figure 1.4 shows such a gamma-ray microscope aimed at an electron. The gamma rays are scattered by the electron into the objective lens of the microscope (in practice, no lenses for gamma rays are available, but we will ignore this petty technicality). According to wave optics, the resolving power of the microscope is

$$\Delta x = \frac{\lambda}{\sin \alpha} \tag{20}$$

where α is the angle subtended by the objective lens (see Fig. 1.4). However, the measurement of position is impossible unless at least one photon strikes the electron. When this happens, the electron acquires a recoil momentum and an uncertainty in the momentum, by the Compton effect. The magnitude of the mo-

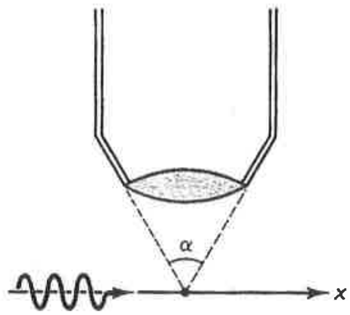


Fig. 1.4 Heisenberg's gamma-ray microscope.

mentum of the scattered photon is h/λ;[2] since the direction of motion of the scattered photon can fall anywhere within the angle α, the horizontal component of the momentum of this photon is uncertain by $\Delta p_x \simeq (h/\lambda)\sin\alpha$, and this must be the uncertainty in the momentum of the electron. The product of the simultaneous uncertainties Δx and Δp_x is then

$$\Delta x\,\Delta p_x = \left(\frac{\lambda}{\sin\alpha}\right)\left(\frac{h}{\lambda}\sin\alpha\right) \simeq h \tag{21}$$

which is, again, consistent with the general uncertainty relation (16).

The uncertainty relation tells us that the position and the momentum of a quantum-mechanical particle are complementary variables; if we perform a measurement that determines one of these with high accuracy, then the other will be poorly determined. Since the momentum is directly related to the wavelength, we can also say that the particle aspect (position) and the wave aspect (wavelength) are complementary. Thus, in any given measurement, either the particle aspect will be displayed or the wave aspect, but not both together. This principle of complementarity was formulated by Bohr.

The following *Gedankenexperiment* neatly illustrates the complementarity of the particle and wave aspects. Consider an opaque plate with two thin, parallel slits and a fluorescent screen placed beyond the plate (see Fig. 1.5). If a beam of monoenergetic electrons is incident on the plate, the electron waves passing through the two slits will form a typical interference pattern on the

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[2] Here, and also in Eq. (20), λ is the wavelength of the photon *after* it is scattered by the electron.

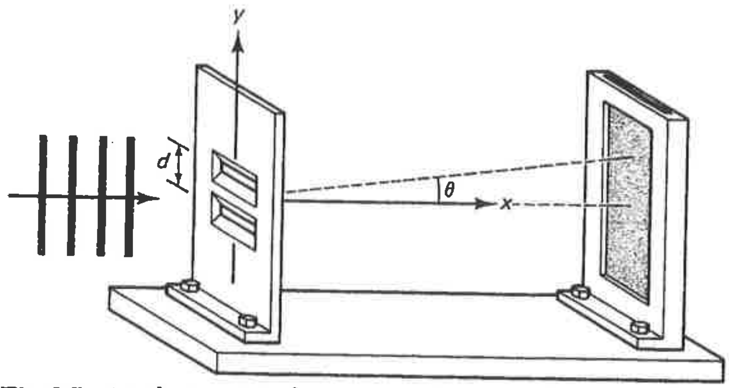

Fig. 1.5 An electron interference experiment involving two slits.

screen. The angular locations of the maxima of this interference pattern are given by a formula familiar from wave optics:

$$d \sin \theta = n\lambda \qquad n = 0, 1, 2, \ldots \tag{22}$$

where d is the distance between the slits. Thus, the experimental arrangement of Fig. 1.5 brings out the wave aspect of the electrons. But suppose we now ask the question: Through which of the two slits does each individual electron pass? The experimental arrangement of Fig. 1.5 does not give us any information on this point, and if we want to investigate the passage of the electrons through the slits, we need some extra experimental equipment, for instance, we might aim a Heisenberg gamma-ray microscope at the slits and watch electrons passing through. Since we want to decide through which of the slits an electron passes, the gamma-ray microscope must measure the position of the electron with an uncertainty no larger than d, that is, $\Delta y \simeq d$. But this implies $\Delta p_y \simeq h/d$, and consequently the direction of motion of the electron acquires an uncertainty.

$$\Delta(\sin \theta) = \frac{\Delta p_y}{p} \simeq \frac{h}{d}\frac{\lambda}{h} \simeq \frac{\lambda}{d} \tag{23}$$

Comparing this with Eq. (22), we see that the uncertainty in the angle is as large as the angular separation between one maximum of the interference pattern and the next. Thus, the interference pattern will be completely washed out—the electrons will strike the fluorescent screen more or less at random. This *Gedankenexperiment* illustrates that we can either let the electrons display their wave properties (if we do not check through which slit the electrons pass) or we can let the electrons display particle properties (if we check through which slit the electrons pass). But if we investigate the particle properties, then the wave properties will remain hidden. Note that what equipment we use to check on the passage of electrons through the slits is irrelevant—any kind of equipment will yield the uncertainty relation $\Delta p_y \simeq h/d$, and therefore lead to the conclusion (23).