Chapter 11
AC circuit Power Analysis

instantaneous vs average power.

11.1 Instantaneous power

$$p(t) = v(t)\, i(t)$$

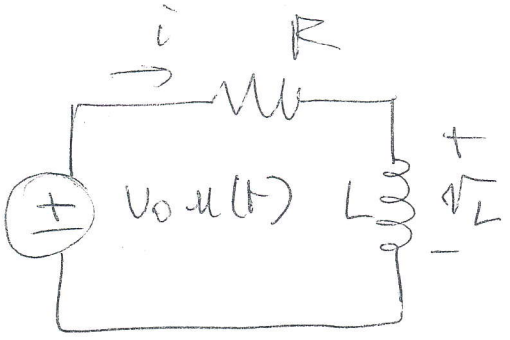
Resistor $\rightarrow$ $p(t) = R\, i^2(t) = \dfrac{v^2(t)}{R}$

Inductor $p = v\, i = L\, i(t) \dfrac{di(t)}{dt} = \dfrac{1}{L} v(t) \int_{-\infty}^{t} v(t)\, dt$

if $v(t=-\infty) = 0$

Capacitor $p(t) = v(t)\, i(t) = C\, v(t) \dfrac{dv(t)}{dt} = \dfrac{i(t)}{C} \int_{-\infty}^{t} i(t)\, dt$

if $i(-\infty) = 0$



$$i(t) = \frac{V_0}{R}\left(1 - e^{-\frac{Rt}{L}}\right) u(t)$$

$$\rightarrow p(t) = v(t)\, i(t) = \frac{V_0^2}{R}\left(1 - e^{-\frac{Rt}{L}}\right) u(t)$$

↑ power delivered by source and absorbed by passive network

Power delivered to resistor

$$P_R(t) = R\, i_R^2(t) = \frac{V_0^2}{R}\left(1 - e^{-\frac{Rt}{L}}\right)^2 u(t)$$

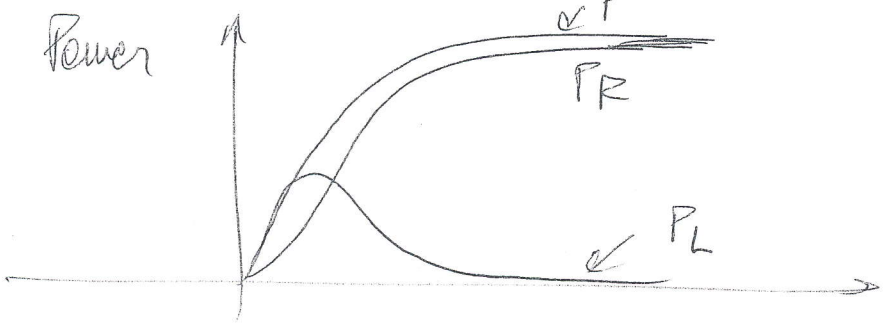
Power absorbed by inductor

$$v_L = L\frac{di}{dt} = V_0 e^{-\frac{Rt}{L}} u(t) + \frac{LV_0}{R}\left(1 - e^{-\frac{Rt}{L}}\right)\underbrace{\frac{du}{dt}}_{=0 \text{ for } t}$$

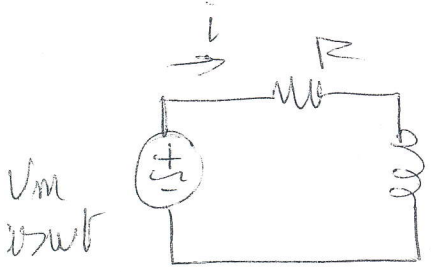
$$= V_0 e^{-\frac{Rt}{L}} u(t)$$

$$\rightarrow P_L = v_L\, i(t) = \frac{V_0^2}{R} e^{-\frac{Rt}{L}}\left(1 - e^{-\frac{Rt}{L}}\right) u(t)$$

and we can check that $p(t) = P_R(t) + P_L(t)$



Power due to sinusoidal excitation



$$i(t) = I_m \cos(\omega t + \phi)$$

$$I_m = \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}}$$

$$\phi = -\tan^{-1}\left(\frac{\omega L}{R}\right)$$

Steady-state

$$p(t) = v(t)\,i(t) = V_m I_m \cos(\omega t + \phi)\cos\omega t$$

$$p(t) = \frac{V_m I_m}{2}\left[\cos(2\omega t + \phi) + \cos\phi\right]$$

$$p(t) = \underbrace{\frac{V_m I_m}{2}\cos\phi}_{\text{average value}} + \frac{V_m I_m}{2}\cos(2\omega t + \phi)$$

11.2 Average power

$$\bar{P} = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} p(t)\,dt$$

Average power of periodic waveforms.

$$f(t) = f(t+T) \qquad T \text{ period}$$

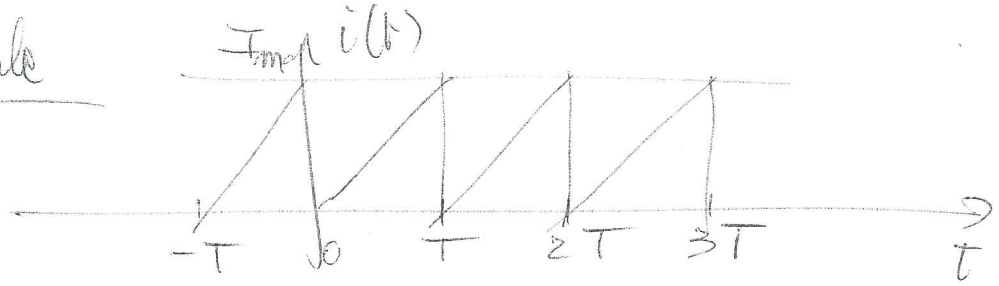
$$P_1 = \frac{1}{T}\int_{t_1}^{t_1+T} p(t)\,dt \qquad P_2 = \frac{1}{T}\int_{t_2}^{t_2+T} p(t)\,dt$$

$$P_1 = P_2$$

$$P_1 = \frac{1}{nT}\int_{t_x}^{t_x+nT} p(t)\,dt = \frac{1}{nT}\int_{-\frac{nT}{2}}^{+\frac{nT}{2}} p(t)\,dt$$

$$\lim_{n\to\infty} \rightarrow \quad P = \lim_{C\to\infty}\int_{-\frac{C}{2}}^{C/2} p(t)\,dt$$

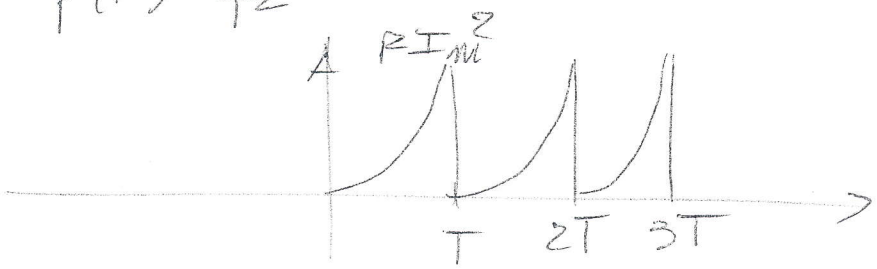
Example



$$i(t) = \frac{I_m}{T} t \qquad 0 < t \le T$$

$$i(t) = \frac{I_m}{T}(t-T) \qquad T < t \le 2T$$

$$\therefore \quad p(t) = \frac{1}{T^2} I_m^2 R t^2 \qquad 0 < t \le T$$



$$P = \frac{1}{T}\int_0^T \frac{I_m^2 R}{T^2} t^2 \, dt = \frac{1}{3} I_m^2 R$$

Average power in sinusoidal steady state

$$v(t) = V_m \cos(\omega t + \theta)$$

$$i(t) = I_m \cos(\omega t + \phi)$$

$$p(t) = I_m V_m \cos(\omega t + \theta)\cos(\omega t + \phi)$$

$$p(t) = \frac{1}{2} V_m I_m \cos(\theta - \phi) + \frac{1}{2} V_m I_m \cos(2\omega t + \theta + \phi)$$

$$\rightarrow P = \frac{1}{2} V_m I_m \cos(\theta - \phi)$$

Example 11.2 If $v = 4\cos\frac{4t}{6}$

Find both the average power and instantaneous power and both results if the corresponding phasor $V = 4\angle 0^\circ$ V is applied across the impedance $Z = 2\angle 60^\circ\ \Omega$

Phasor current $V/Z = 2\angle -60^\circ$ A

$$\rightarrow P = \frac{1}{2} 4 \cdot 2 \cdot \cos(0 - (-60^\circ)) = 2\ W$$

$$v(t) = 4\cos\frac{4}{6}t \rightarrow i(t) = 2\cos\left(\frac{4t}{6} - 60^\circ\right)$$

$$\rightarrow p(t) = 8\cos\frac{4}{6}t\cos\left(\frac{4t}{6} - 60^\circ\right)$$

$$\boxed{p(t) = 2 + 4\cos\left(\frac{4t}{3} - 60^\circ\right)\ W}$$

Average power absorbed by an ideal resistor

$$P_R = \frac{1}{2} V_m I_m \cos\theta = \frac{1}{2} R I_m^2 = \frac{V_m^2}{2R}$$

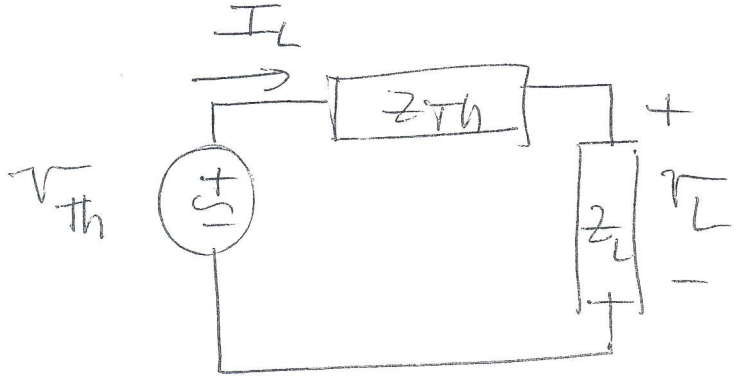
Average power absorbed by purely reactive elements

$$\cos(\theta - \phi) = \cos \pm 90^\circ = 0$$

$$\rightarrow \boxed{P_X = 0}$$

Maximum Power Transfer

An independent voltage source in series with an impedance Z_{Th} or an independent current source in parallel with an impedance Z_{Th} delivers a maximum average power to a load Z_L which is the conjugate of Z_{Th}, or $Z_L = Z_{Th}^*$



$$Z_{Th} = R_{Th} + j X_{Th}$$

$$Z_L = R_L + j X_L$$

Current through loop

$$I_L = \frac{V_{Th}}{Z_{Th} + Z_L} = \frac{V_{Th}}{R_{Th} + R_L + j(X_{Th} + X_L)}$$

$$V_L = V_{Th} \frac{Z_L}{Z_{Th} + Z_L} = V_{Th} \frac{R_L + j X_L}{\left[R_{Th} + R_L + j(X_{Th} + X_L)\right]}$$

$$|I_L| = \frac{|V_{Th}|}{\sqrt{(R_{Th} + R_L)^2 + (X_{Th} + X_L)^2}}$$

Phase angle $\quad \angle V_{Th} - \tan^{-1}\left(\frac{X_{Th} + X_L}{R_{Th} + R_L}\right)$

Magnitude of $|V_L|$

$$\frac{|V_{Th}| \sqrt{R_L^2 + R_X^2}}{\sqrt{(R_{Th} + R_L)^2 + (X_{Th} + X_L)^2}}$$

Phase angle

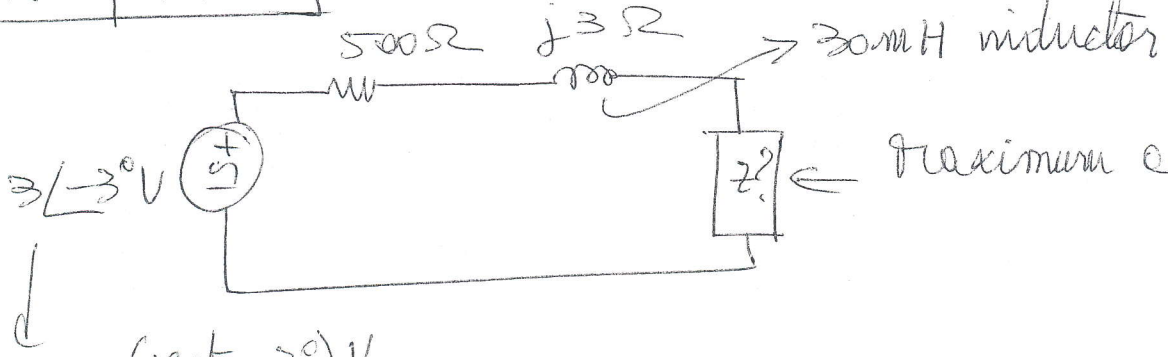
$$\angle V_{Th} + \tan^{-1}\left(\frac{X_L}{R_L}\right) - \tan^{-1}\left(\frac{X_{Th} + X_L}{R_{Th} + R_L}\right)$$

Average power

$$P = \frac{\frac{1}{2}|V_{Th}|^2 \sqrt{R_L^2 + X_L^2}}{(R_{Th}+R_L)^2 + (X_{Th}+X_L)^2} \cos\left(\tan^{-1}\left(\frac{X_L}{R_L}\right)\right)$$

$$\frac{\partial P}{\partial R_L} = 0 \qquad \frac{\partial P}{\partial X_L} = 0 \quad \Longrightarrow \quad \boxed{Z_L = \overline{Z}_{Th}}$$

Example 11.5



$3\cos(100t - 3°)$ V

$$Z_? = \overline{Z}_{Th} = 500 - j3\,\Omega$$

↳ 500-Ω resistor in series with a capacitor of impedance $Z_C = \frac{1}{j\omega C} = \frac{1}{j100C} = -j3\,\Omega$

$$\frac{1}{100C} = 3 \rightarrow \boxed{C = \frac{1}{300} = 3.33\,\text{mF}}$$