Code Number: _____

UC-CS/ECE-DQE

21 September 2007

Math Foundations I (Mostly Induction). (**Warning:** Write carefully; this problem's grader is very concerned about the *forms* of inductive proofs. For example, be sure to state inductive hypotheses very precisely.)

1. **[25 pts]**

   (a) Which of the following numbers is largest? (a) 9^{9^9}, (b) $9!!$, (c) $9^9!$
   Justify your answer. (Use the usual priority and associativity assumptions: $9^{9^9} = 9^{(9^9)}$, $9!! = (9!)!$, and $9^9! = (9^9)!$.)

   (b) Which of the following 3 numbers is largest?

   $$(\mathrm{a})\ 9^{9^{9^{9^{9^{9^{9^{9^{9}}}}}}}} \qquad (\mathrm{b})\ 9!!!!!!!! \qquad (\mathrm{c})\ 9^9!!!!!!!$$

   Prove your answer, using your answer from part (1a). Note that (a), (b), and (c) is each written with 9 symbols. *For full credit, you must state a generalization of this result and prove it by induction on the number of symbols.*

2. **[25 pts]** Below we illustrate *pairing parentheses* in mathematical formula:

   $$\underbrace{(3.0+x)} \cdot \underbrace{(\ \underbrace{(2.0-x^2)}\ /\ (\ x+\ \underbrace{(2.0-x)}^2\ )\ )}$$

   The left parenthesis — '(' — at the start of each "underbrace" is paired with the right parenthesis — ')' — at its end. A pairing (or *proper pairing*) of parentheses is a 1-1 correspondence between left parentheses and right parentheses in the formula such that

   (a) every left parenthesis is paired with a right parenthesis to the right of it, and
   (b) if we have paired two pairs of parentheses in a single expression, they must be paired as either

   $$\cdots\underbrace{(\cdots\underbrace{(\cdots)}\cdots)}\cdots \qquad \text{or} \qquad \cdots\underbrace{(\cdots)}\cdots\underbrace{(\cdots)}\cdots$$

   — one pair must be entirely within the other, or one must come entirely after the other.

   Prove that the parentheses in a string σ of characters can be (properly) paired if and only if:

   (a) σ has the same number of right parentheses as left parentheses, and
   (b) for τ any initial substring of σ (*i.e.*, whenever $\sigma = \tau\upsilon$ for some string υ), τ has at least as many left parentheses as right parentheses.

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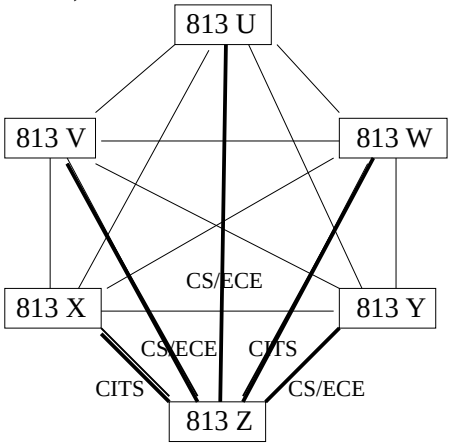
Math Foundations II (Mostly Counting). Parts (1-2) require only answers, but, to qualify for partial credit for wrong answers, you may have to clarify how you got them.

1. **[21 pts]** A design is made by placing 8 stones on the 144 squares of a 12×12 square checkerboard. Each stone occupies just 1 square, and 2 stones may not occupy the same square. Assume we can't tell the stones apart, so we don't count interchanging 2 stones as giving us a different design. (But we *can* tell all the squares of the board apart — *e.g.*, they might be numbered 1-144.) *Do not simplify your answers to this question.*
   (a) How many different designs can be made?
   (b) How many different designs can be made with ≤ 1 stone in each row?
   (c) How many different designs can be made with ≤ 1 stone in each row and ≤ 1 stone in each column?

2. **[7 pts]** How many strictly monotonically increasing functions are there from the set $\{1, 2\}$ to the set $\{1, 2, \ldots, 1000\}$ (the set of all integers from 1 to 1000)? *Simplify your answer.*

3. **[22 pts]** To preserve maximum communications privacy, UC has installed private network lines linking each pair among 6 mailboxes in the CS/ECE mailroom (813 Rhodes). The lines are shown below, with the lines connecting to 813Z shown in boldface. Some of the lines are maintained by CITS; and the rest, by the CS/ECE departments.
   (a) Suppose that the groups maintaining the lines out of 813Z are as shown below (and we don't know who maintains the others).



   Prove that there must be 3 mailboxes where (i) all 3 lines connecting the 3 to each other are controlled by CITS or (ii) all 3 lines connecting the 3 to each other are controlled by CS/ECE.

   (b) Prove that, no matter which organization (CITS or CS/ECE) manages each of the 15 communications lines, there will be 3 mailboxes where all 3 lines connecting them to each other are maintained by the same organization.