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Using chaos to produce synchronized stochastic dynamics in non-homogeneous map arrays with a random scalar coupling

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Abstract

Unidirectionally coupled chaotic systems hold great interest from the information processing and communications perspective. In this Letter, we report on a novel method for synchronizing two identical but internally non-homogeneous populations of chaotic maps using a scalar random coupling between them. The resulting synchronized dynamics is stochastic, and can be used in secure multi-user communication applications. © 1999 Published by Elsevier Science B.V.

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1. Introduction

Synchronization between unidirectionally coupled chaotic systems [1,2] is of considerable interest from the theoretical and applied standpoint. In particular, it has been suggested that such systems may be useful for secure communications [3–8]. In most such proposals, the chaotic coupling signal is used as the carrier of information – either directly or through modulation – and synchronization between chaotic systems in the transmitter and receiver allows the recovery of the message. While there are several impediments to such applications, two key issues of interest are: (1) Making the coupling signal maximally unstructured [9], so that the hidden message cannot be recovered through reconstruction and noise-removal methods [10–12]; and (2) multiplexing several messages onto the same scalar coupling signal [13]. In this Letter, we report on a novel synchronization technique that is relevant to both issues. In particular, we show that identical but internally inhomogeneous arrays of chaotic maps can be synchronized if coupled through a *random scalar signal* from the drive array to the response array. As a result, corresponding maps in each array generate synchronized high-dimensional *noise* rather than low-dimensional chaos, though chaos is essential for the synchronization to emerge. Thus, the method we present contrasts with all other schemes for chaotic synchronization, where the coupling is deterministic – though perhaps intermittent [14] – and the synchronized signals are deterministic – albeit chaotic. The availability of truly random synchronized signals has obvious utility in real-time encrypted communication, where the signals can be used as encryption keys [15,16,8]. The method we report is also applicable to secure multi-user spread-spectrum digital communication [17,18], with the random signals generated by individual maps used as aperiodic spreading

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sequences. The random nature of the scalar coupling signal makes it virtually impossible for an intruder to extract the messages it carries. Detailed description and performance analysis for these applications will, however, be presented elsewhere. In the interest of brevity, this Letter focuses only on the synchronization mechanism.

2. System description and synchronization mechanism

Synchronization in unidirectionally coupled maps has recently been studied by several researchers [13,7,19–21,9]. Mozdy et al. [22] have also shown synchronization in unidirectionally coupled chains of chaotic diode resonators. In most cases, synchronization is obtained through explicit or implicit differential feedback control, though more general methods have also been applied [21]. One key requirement of all these methods is that a very specific signal be communicated from the drive system to the response system. This greatly constrains the coupling and makes the synchronization of high-dimensional systems more difficult [7,13]. In recent reports [23,24], we have shown that identical chaotic maps can, in some cases, be synchronized by a common noise-like input, which must only meet certain relatively mild statistical constraints [25]. In this Letter, we show that the same effect can be used to synchronize large identical populations of different maps coupled by a scalar signal which is a random function of the states of the driving system maps. Thus, the coupling signal is not just noise-like; it *is* noise, albeit constructed via the random sampling of a large number of different low-dimensional chaotic signals. The coupling between the drive and response systems is not diffusive, and synchronization occurs through coalescence [26–28,23,24] rather than differential feedback.

The map we use is based on a discrete-time neural oscillator studied by Wang [29],

$$z_{t+1} = f(z_t, u_t) = \tanh(\mu A z_t + u_t) - \tanh(\mu B z_t), \quad (1)$$

where $A, B > 0$, $A/B \geqslant 2$ and u_t is an external input. If μ is large enough, the map is chaotic for $u_t = 0\ \forall t$ [29]. For such a chaotic map, using a fixed input $u_t = u > 0\ \forall t$ changes the dynamics, producing period-halving with increasing u, culminating in an extensive period-2 regime [23,30]. Fig. 1 shows the bifurcation diagram and the corresponding Lyapunov exponent, $\lambda(u)$, for the map with $\mu = 5.0$, $A = 5.0$, and $B = 1.0$.

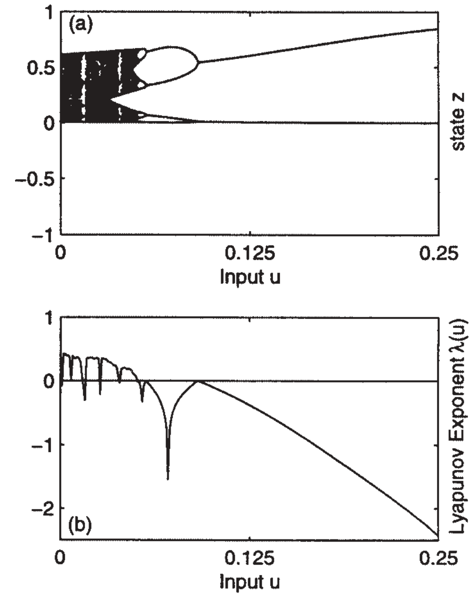

Fig. 1. Bifurcation diagram and Lyapunov exponents for the map in Eq. (1) with respect to a fixed input, u. Map parameters are $\mu = 5$, $A = 5$, and $B = 1$. The Lyapunov exponent is calculated numerically.

Two identical maps of this type, if coupled unidirectionally in a diffusive way, will synchronize [31]. However, the extension of this methods to multiple maps requires careful choice of parameters [13,7] – especially when individual map pairs are different. We show, however, that map populations coupled non-diffusively can be synchronized with only minimal tuning of the coupling parameter(s). This synchronization is very rapid and stable if the system is implemented with finite precision, e.g., in digital hardware or as software on a computer, as most encryption, signal-processing and digital communication systems are. In physical systems such as analog circuits [22] or lasers, the synchronization would be intermittent.

Consider two identical populations of maps, $P^d = \{z_t^{d_k}\}$ and $P^r = \{z_t^{r_k}\}$, where $k = 1, \ldots, N$ indexes the individual map pairs, and d and r denote the *drive* and *response* systems, respectively. The system is shown

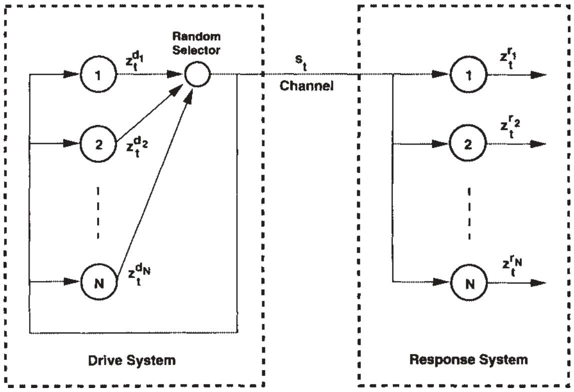

Fig. 2. System architecture: Each circle represents a map, $f_k(\ )$. Identically numbered maps in the drive and response systems are identical, but maps within each system may be different (but within the chaotic regime). The coupling signal, s_t, is produced by the selector switch by random multiplexing of the $z_t^{d_k}$ signals.

in Fig. 2. The maps within each population may all be different, making it an inhomogeneous population. However, corresponding maps across the populations are identical. We assume that parameters are set such that all maps are intrinsically chaotic. The synchronizing signal is chosen to be

$$s_t = g(z_t^{d_1}, \dots, z_t^{d_N}) = z_t^{d_{q_t}},$$

where $q_t \in \{1, \dots, N\}$ is chosen randomly at each step with a uniform distribution. This signal is provided as input to each map. Thus, each population is a *globally coupled map network with sparse random time-varying connectivity and non-identical elements.* This contrasts with previous studies of synchronization in coupled map lattices of identical elements (see, e.g., Refs. [32,7,33]).

The equations for the kth map pair are

$$\begin{aligned} z_{t+1}^{d_k} &= f_k(z_t^{d_k}, \alpha_k s_t) \\ &= \tanh[\mu_k(A_k z_t^{d_k} + \alpha_k s_t)] - \tanh(\mu_k B_k z_t^{d_k}), \\ z_{t+1}^{r_k} &= f_k(z_t^{r_k}, \alpha_k s_t) \\ &= \tanh[\mu_k(A_k z_t^{r_k} + \alpha_k s_t)] - \tanh(\mu_k B_k z_t^{r_k}). \end{aligned} \tag{2}$$

The coupling term, $s_t = z_t^{d_{q_t}}$ is, therefore, a random function of the states of all maps in P^d, and α_k is the coupling strength for the kth map pair. As N becomes large, s_t has characteristics approaching very high-dimensional noise.

The stability of synchronization in chaotic systems is often studied by evaluating the transverse conditional Lyapunov exponents (CLE's) of the differential dynamics along the synchronized trajectory [2]. The kth difference variable is defined as $e_t^k = z_t^{d_k} - z_t^{r_k}$, and its dynamics along the synchronized trajectory are given by

$$\begin{aligned} e_{t+1}^k &= F_k(e_t^k, z_t^d, s_t) \\ &= \tanh[\mu_k(A_k z_t^d + \alpha_k s_t)] - \tanh(\mu_k B_k z_t^d) \\ &\quad - \tanh[\mu_k(A_k(z_t^d - e_t^k) + \alpha_k s_t)] \\ &\quad + \tanh[\mu_k B_k(z_t^d - e_t^k)]. \end{aligned}$$

The N-dimensional error vector is $e_t = [e_t^1 \dots e_t^N]$. The $e_t = 0$ hyperplane is then an invariant manifold for the dynamics. To analyze the stability of $e_t = 0$, we consider the kth pair of maps in the two arrays. These can be seen as two identical, uncoupled maps driven by the "external" random signal, s_t. The conditional Lyapunov exponent for the synchronized trajectory $e_t^k = 0$ is, then,

$$\begin{aligned} \Lambda &= \left\langle \ln \left| \partial F(e_t, z_t^{d_k}, s_t) / \partial e_t^k \right| \right\rangle \\ &= \langle \ln | f_k'(z, \alpha_k s_t) | \rangle \\ &= \lim_{T \to \infty} \frac{1}{T} \sum_{t=\tau}^{\tau+T-1} \ln | f_k'(z_t^{d_k}, \alpha_k s_t) |, \end{aligned} \tag{3}$$

where the partial derivative is evaluated along $e_t^k = 0$, $\langle\ \rangle$ indicates averaging over the invariant distribution of $(z_t^{d_k}, s_t)$, and $f_k'(z, s) = \partial f_k(z, s)/\partial z$. If s_t is independent of $z_t^{d_k}$, the CLE of the synchronized trajectory can be approximated by $\Lambda = \int_s \rho(s) \lambda(\alpha_k s)\, ds$, where $\lambda(\alpha_k s)$ is the Lyapunov exponent of the $f_k(\ )$ map with a fixed input $s_t = s\ \forall t$, and $\rho(s)$ is the stationary distribution of s_t. Of course, s_t and $z_t^{d_k}$ are not actually independent, but the approximation is appropriate for large N. From the typical Lyapunov exponent plot shown in Fig. 1b, it is obvious that many input distributions, $\rho(s)$, can make Λ negative and stabilize the $e_t^k = 0$ trajectory. For synchronization to actually arise between the maps, s_t must also satisfy the condition $\int_0^{s^*} \rho(s)\, ds > 0$, where s^* is the largest input such that the map $f_k(z, \alpha_k s)$ is in the single-band chaotic regime. This, in conjunction with $\Lambda < 0$, ensures that

$e_t^k = 0$ is the only invariant manifold in $(z_t^{d_k}, z_t^{r_k})$-space and it is attracting on average.

As discussed by several researchers [34–37], a negative CLE only implies that the synchronized trajectory is attracting on average, since the CLE is defined in the limit of infinite time. Desynchronization episodes of finite duration are possible, and depend on finite-time quantities called local Lyapunov exponents [38,39,34,35]. However, if the system is implemented with finite precision (e.g., in a digital system or in software), a negative CLE ensures that synchronization stabilizes within a short time. We have shown elsewhere [31,25] that, beginning with $|e_0^k| \sim O(1)$, the time, t_ϵ, for the trajectories to come within $|e_t^k| \leqslant \epsilon$ scales as $t_\epsilon \sim \log(1/\epsilon)$ (when the CLE is positive, $t_\epsilon \sim 1/\epsilon$ [28]). Since many applications of synchronizable maps – notably in encryption and digital communication – use finite-precision, coalescence-based synchronization is a potentially useful phenomenon.

One notable point about the system we describe is that, while the Lyapunov exponents of each map *conditioned on the driving input* are negative by construction, the dynamics of z^{d_k} and z^{r_k} is aperiodic and random due to the randomness of the driving signal. This is analogous to the original Pecora–Carroll synchronization method [1,2], where an intrinsically non-chaotic subsystem is slaved to a chaotic driving signal.

3. Numerical results

To demonstrate that the synchronization implied by the above analysis actually emerges, we numerically investigate the synchronization of a specific 10-map system with $A_k = 4.0 + 0.5(k-1)$ and $B_k = 1.0$, while μ and α are varied systematically over the 0.0–10.0 and 0.0–1.0 ranges, respectively. The values of μ and α for all maps are identical in any given run. Fig. 3 shows the domain of reliable synchronization for this system. Results for other systems are qualitatively the same. The simulation used a numerical precision of 10^{-16}.

We have also investigated the effect of additive noise in s_t to determine whether synchronization persists in that situation. We find that the effect of additive noise is to produce intermittent desynchronization, called "attractor bubbling" [40], which indicates

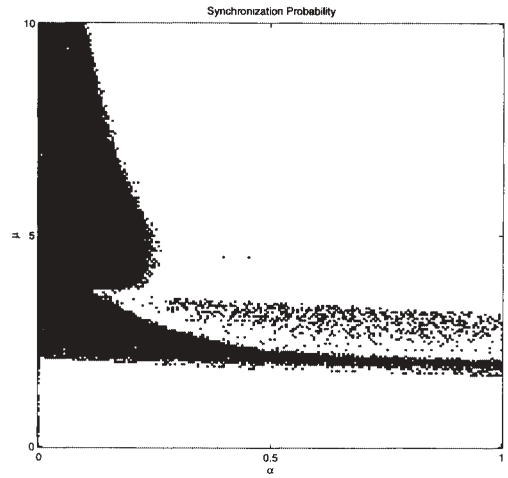

Fig. 3. Synchronization probability for a 10-map system, f_k, $k = 1, \ldots, 10$, with $A_k = 4.0 + 0.5(k-1)$ and $B_k = 1.0$, and different global settings of μ and α. White indicates synchronization with probability 1 over 20 independent runs. The white strip for low μ values shows trivial synchronization due to all maps converging to fixed points.

that the synchronized trajectory is the only attractor for the system. This is confirmed by simulations of the system from a very large number (8×10^6) of initial conditions near the invariant manifold with μ and α set in the synchronization regime. All initial conditions are attracted to the invariant manifold (results not shown).

To demonstrate that the method is applicable to much larger systems without detailed tuning, we simulated the case where the drive and response systems consist of 100 maps each. Each map is chosen randomly by setting the μ_k, A_k, and B_k parameters to random values within broad ranges. The corresponding maps in the two subsystems are, of course, identical. All maps have the same coupling strength, α. Fig. 4 shows the empirical probability and mean time of synchronization as a function of α, with each point averaged over 50 different random systems and initial conditions. It is clear that, beyond a threshold value, any setting for α will reliably synchronize any system within the parameter ranges. To demonstrate this, we simulated several randomly generated systems of 100-map drive and response subsystems where each map pair had a randomly chosen α between 0.7 and 1.0. Synchronization was obtained in every case, showing that no specific tuning is necessary for any system pa-

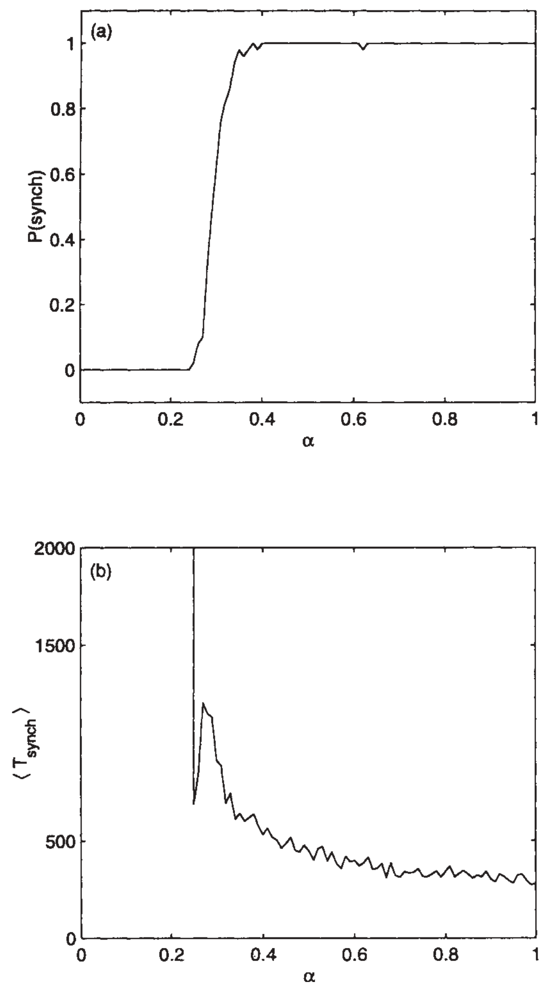

Fig. 4. Graph (a) shows the probability of synchronization and Graph (b) the mean synchronization time as a function of α for a 100-map system with parameters set as $\mu_k \sim U[5.0, 10.0]$, $A_k \sim U[5.0, 10.0]$, and $B_k \sim U[1.0, 1.5]$, where $U[a, b]$ denotes a uniform random variable between a and b. All map pairs have the same α. The data for each α is averaged over 50 independent runs.

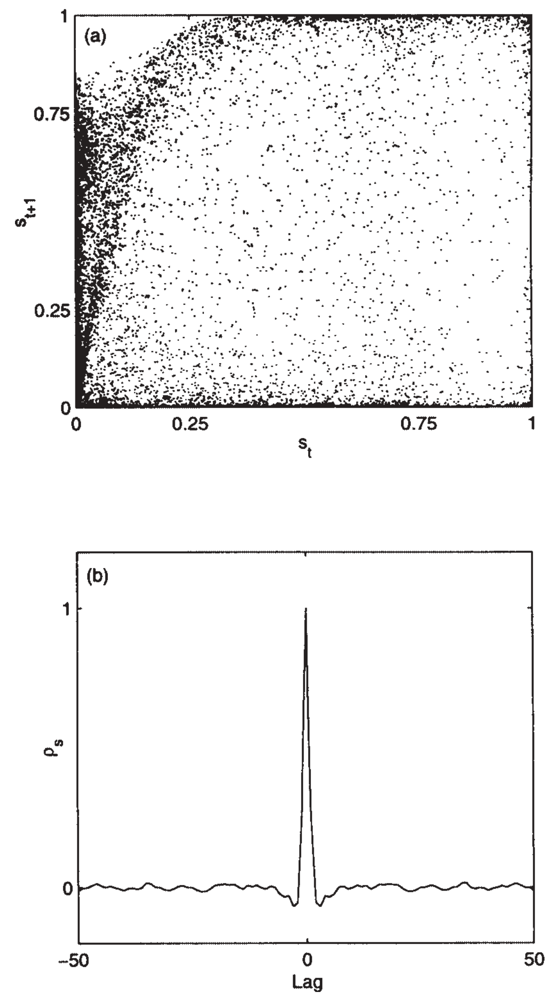

Fig. 5. Graph (a) shows the one-step return map for 20,000 points of s_t in a 100-map system, f_k, $k = 1, \ldots, 100$, with $\mu_k \sim U[5.0, 10.0]$, $A_k \sim U[5.0, 10.0]$, $B_k \sim U[1.0, 1.5]$, and $\alpha_k \sim U[0.7, 1.0]$. Graph (b) shows the normalized autocorrelation function, ρ_s, for the centered signal $s_t - \langle s_t \rangle$.

rameter provided they fall into certain broad ranges. Fig. 5 shows the return map and autocorrelation function of the coupling signal s_t for one of the 100-map systems, and the space-filling nature of the signal is quite apparent. The residual structure is due to the fact that the $z_t^{d_k}$ variables from which s_t is constructed have invariant distributions that are far from uniform.

The noise-like nature of the coupling signal is potentially useful for secure communications applications. When standard chaotic signals are used as carriers to mask messages [3,41], reconstruction methods such as delay-coordinate embedding can be used to extract the hidden message [10–12]. Thus, there has been a search for more high-dimensional and noise-like carriers [7,9] which are less vulnerable to reconstruction. The coupling signal in our system is, in fact, random but is still capable of carrying information (since it is derived – albeit randomly – from the drive maps).

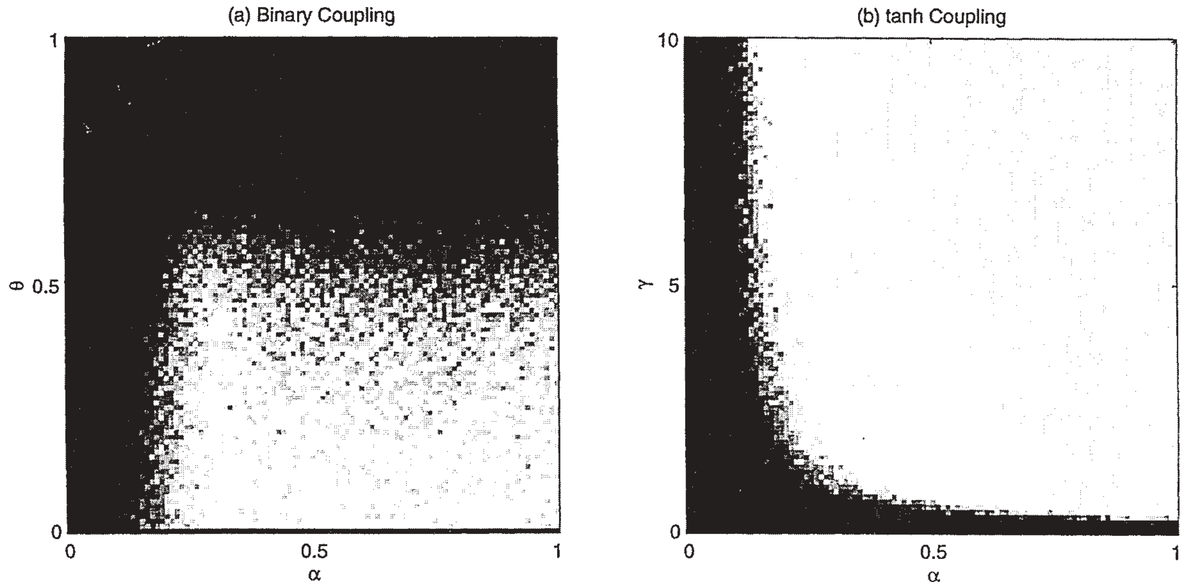

Fig. 6. Probability of synchronization in random 10-map systems, f_k, $k = 1, \ldots, 10$, with binary coupling (a), and a hyperbolic tangent coupling (b). Each data point is averaged over 10 independent runs, with the system parameters set randomly as $\mu_k \sim U[5.0, 10.0]$, $A_k \sim U[5.0, 10.0]$, and $B_k \sim U[1.0, 1.5]$. White indicates probability 1, and black probability 0.

4. Transformed coupling signals

In some situations, it may be desirable to use a transformed version of s_t for coupling, and this can be done if the transformation chosen is appropriate. The coupling signal, communicated to both P^d and P^r, is then $s'_t = h(s_t)$, where $h(x)$ is the transformation. If $h(x)$ approximates the actual cumulative distribution function (CDF) of s_t, the transmitted signal, s'_t will, in fact, be uniformly distributed. Here, we show numerical results for two simple transformations. In the first case, we define

$$s'_t = 0 \quad \text{if} \quad s_t < \theta,$$
$$= 1 \quad \text{if} \quad s_t \geqslant \theta,$$

where θ is a threshold parameter chosen by the user. The coupling signal, s'_t, is now a random binary signal which is multiplied by α_k when input to the kth map pair. We have shown previously [23] that a random telegraph signal (RTS) can synchronize the maps of type (1). However, the range of α within which the synchronization occurs depends on μ, A, and B. Fig. 6a shows the synchronization probability in α–θ space for randomly generated arrays of 10 maps. A large domain of certain or near-certain synchronization is apparent. The reliability of synchronization in this domain can be improved further by restricting the range of map parameters and α more carefully. A thresholded coupling signal has several advantages, including higher noise immunity and less information about the system which generates the underlying s_t. The latter attribute could be especially helpful in secure communications applications.

The second case we consider is to define the coupling signal by $s'_t = \tanh(\gamma s_t)$, where γ is a user-defined gain parameter. This type of coupling may be relevant when the maps are used as models of neural assemblies [42]. As shown in Fig. 6b, a large domain of synchronization is again obtained in the α–γ space. We also note that in cases where $h(x)$ is invertible, it is possible to use s'_t for coupling but the original s_t signal for driving the maps.

5. Discussion

An interesting point about the method we describe is that it requires *multiplicity* and *diversity* of maps in each population. A system with $N = 1$ will not synchronize, since the coupling signal in this case is simply the output of the drive map, which leads to the drive and response system equations

$$z_{t+1}^d = \tanh[\mu(A+\alpha)z_t^d] - \tanh(\mu B z_t^d),$$
$$z_{t+1}^r = \tanh[\mu(Az_t^r + \alpha z_t^d)] - \tanh(\mu B z_t^r) \qquad (4)$$

effectively makes the drive and response maps non-identical and not subject to coalescence-based synchronization [23,24]. However, higher-dimensional maps constructed from $f(z_t)$, can be synchronized in this way [8]. Also, the system will not synchronize reliably if all of the maps within P^d and P^r are identical. In this case, the issue is whether all drive system maps synchronize among themselves before they can synchronize the corresponding response maps. If synchronization within P^d occurs first, the drive system reduces to an $N = 1$ system, precluding synchronization between P^d and P^r. If not, the drive and response systems may synchronize. However, the uncertainty of such synchronization renders it less useful.

6. Conclusion

The method we have presented differs from existing methods for synchronizing high-dimensional chaotic systems, and has several unusual features: (1) It uses a random function of the drive system variables rather than a deterministic signal. Thus, any single map in the response population only receives input from its own drive counterpart sporadically, but *does* receive input from other maps the rest of the time, making it very different from schemes based on impulsive driving [14]. (2) The synchronization is based on coalescence rather than differential feedback control, which makes the process relatively insensitive to the precise coupling signal as long as it is sufficiently strong and sufficiently random [23–25]. (3) The synchronized dynamics that results is random rather than just chaotic, making its reconstruction extremely difficult for an intruder. (4) The systems within each chaotic populations are not identical. This is partly what allows the coupling signal to become truly complex and very difficult to reconstruct, and raises the possibility of multiplexed communication over the single-coupling channel. A method for synchronizing groups of non-identical continuous-time systems has been reported in Ref. [43].

In conclusion, we have demonstrated an extremely simple scheme for synchronizing unidirectionally coupled chaotic maps, and shown how it can be used to synchronize groups of different maps through a scalar coupling signal constructed randomly from the map states. The coupling signal can carry a large number of multiplexed messages but has characteristics approximating white noise.

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