Math Logic, Week#XI

Moon's Day

Review Questions (for me to ask & you to answer)**:**

1. We showed that $Th(\mathfrak{N}_S)$ admits elimination of quantifiers.
   ¿What does it mean that a theory admits elimination of quantifiers?
2. **¿What is a Σ_1 formula?**
3. **¿What is a recursive function (as defined in this class)?**
4. **¡Prove that, if the graph of a function f is r.e., it is recursive!**
5. **¿What does the notation $\underline{\mu\, x_{k+1}\, R(x_1, \ldots, x_k, x_{k+1})}$ mean?**
6. **¿What was our theorem about the μ operator and recursive functions?**
7. **¿How did we code sequences of integers by integers?**
8. **¡Name some functions on those sequences that are recursive!**
9. **¿How did we encode formulas with integers?**
10. **¿Is the set of (Gödel numbers of) consequences of a finite set of axioms recursive?**
11. **¿What does it mean that a relation on $\mathbb{N}$ is representable in $\mathbb{A}_E$?**
12. **¿What do we know about the structure of models of $\mathbb{A}_E$?**

More on Representability:

1. **Analogue of Homomorphism Theorem:**
   Suppose $\mathfrak{A} \models \mathbb{A}_E$, $\mathcal{S}t$ is its standard part, and $s : V \to \mathcal{S}t$:
   - *For any Δ_0 formula α,*
   $$\mathfrak{A} \models \alpha[s] \quad \textit{iff} \quad \mathfrak{S}t \models \alpha[s].$$
   - *For any Σ_1 formula $\exists x \alpha$,*
   $$\textit{if } \mathfrak{A} \models \exists x \alpha \textit{ then } \mathcal{S}t \models \exists x \alpha.$$

   (Proof by induction on formulas; similar to homomorphism theorem)

2. **And we know that** (i) $\mathfrak{N}_E \cong \mathcal{S}t$, and every element of $\mathfrak{N}_E$ — and every element of $\mathcal{S}t$ — is named by a term $\underline{k} = S^k(0)$.
   So we can think of any $\mathfrak{A} \models \mathbb{A}_E$ as having $\mathfrak{N}_E$ as its standard part.

3. **Corollary:** *Every Δ_0-definable relation on $\mathbb{N}$ is representable in* $\mathbb{A}_E$ — and representable by the same Δ_0 formula.

4. **What we'd like: that every recursive relation is represntable.**
   - **This is trickier:** Consider a Σ_1 definition of R —
   $$\exists x \varphi(x, v_1, \ldots, v_k).$$
   - If $(v_1, \ldots, v_k) \in R$, then there's a "witness" x in $\mathcal{S}t$. ¡OK!
   - But we know almost nothing about what happens outside the standard part of $\mathfrak{A}$.
     So, out there, there might be such a witness x outside $\mathcal{S}t$ —- creating some "false" elements of R.
   - **¡We need a way around that!**

5. **Thrm:** Every recursive relation $R \subseteq \mathbb{N}^k$ is representable in $\mathbb{A}_E$.

   Proof: If R is recursive, both it and its complement have Σ_1 definitions:

$$\begin{array}{rcl} R(y_1, \ldots, y_k) & \textit{iff} & \mathfrak{N}_E \models \exists x \phi(x, y_1, \ldots, y_k) \\ \overline{R(\vec{n})} & \textit{iff} & \mathfrak{N}_E \models \exists x \psi(x, y_1, \ldots, y_k) \end{array}$$

   Then we have the following:

   (a) For each choice of $y_1, \ldots, y_k$ in $\mathbb{N}$, exactly one such x exists in $\mathfrak{N}_E$ —
   and thus in the starndard part of any $\mathfrak{A} \models \mathbb{A}_E$.

   (b) For all we know, there might be other values of x in the non-standard part of $\mathfrak{A}$ that satisfies either (or even both) of those two formulas.
   <u>But</u>, in $\mathfrak{A}$, it's $> \underline{k}$.

   (c) So we need a formula to pick out the *least* such "witness" x:

$$\begin{array}{rcl} R(\vec{n}) & \textit{iff} & \exists x(\varphi(x, \vec{n}) \\ & & \quad \wedge \forall y_{\leq x} \neg \psi(y, \vec{n}) \\ & &) \end{array}$$

Homework Problem E:

1. Suppose R is a recursive set (unary relation) and that $f_1, \ldots, f_n$ are Σ_1-definable functions. (The f_i's may well have different numbers of argument places; say that f_i is k_i-ary.) The *closure* of R under the f_i's is the smallest set S such that (1) $R \subseteq S$ and (2) for each f_i and each $x_1, \ldots, x_{k_i} \in S$, $f_i(x_1, \ldots, x_{k_i}) \in S$. This set S can be constructed recursively as follows:

$$\begin{aligned} S_0 &= R \\ S_{i+1} &= S_i \cup \bigcup_{1 \leq i \leq n} \{f_i(x_1, \ldots, x_{k_i}) : x_1, \ldots, x_{k_i} \in S_i\} \\ S &= \bigcup_{i \in \mathbb{N}} S_i \end{aligned}$$

   Prove that S is Σ_1-definable (i.e., r.e.).

2. Suppose that, in addition, the functions $f_1, \ldots, f_n$ are *increasing*; that is,
   whenever $f_i(x_1, \ldots, x_{k_i}) = y$, $y > x_1$, $y > x_2$, ..., and $y > x_{k_i}$. Prove that the set S above is then recursive.

Incompleteness:

1. In this section we continue to assume that we are working in the language of $\mathfrak{N}_E$. Generalizations are possible.
2. **Fixed Point Theorem:** For any formula β in which only v_1 occurs free, there is a sentence σ such that

$$\mathbb{A}_E \models (\sigma \leftrightarrow \beta(S^{\#(\sigma)}0)).$$

Discussion, quoted from Enderton, page 227:

> *We can think of σ as* indirectly *saying, "β is true of me." Actually σ doesn't say anything of course, it's just a string of symbols. And even when translated into English according to the intended structure $\mathfrak{N}_E$, it then talks of numbers and their successors and products and so forth. It is only by virtue of our having associated numbers with expressions that we can think of σ as referring to a formula, in this case to σ itself.*

Proof: **[slightly expanded from Enderton]**

Let $\theta(v_1, v_2, v_3)$ be a formula representing the function

$$sb(v_1, v_2) = \begin{cases} \#(\alpha(S^{v_2}0)) & \text{if } v_1 = \#(\alpha) \text{ for some } \alpha \\ & \text{with (at most) 1 free variable} \\ 0 & \text{otherwise.} \end{cases}$$

Trick #1: Consider the formula

$$\forall v_3(\theta(v_1, v_1, v_3) \rightarrow \beta(v_3)). \tag{1}$$

Note that we have set the first two argument positions of θ to the same variable. [So, essentially,] the Gödel-number v_1 of a formula is being substituted into that formula, a process called "self reference." Enderton points out that "*This formula has only v_1 free. It defines in $\mathfrak{N}_E$ a set to which $\#(\alpha)$ belongs iff $\#(\alpha(S^{\#(\alpha)}0))$ is in the set defined by β .*"

Let q be the Gödel-number of formula (1).

Reminders:

- $\theta(v_1, v_2, v_3)$ "says" $$v_3 = \begin{cases} \#(\alpha(S^{v_2}0)) & \text{if } v_1 = \#(\alpha) \text{ for some } \alpha \\ & \text{with (at most) 1 free variable} \\ 0 & \text{otherwise.} \end{cases}$$
- q is the Gödel number of $\forall v_3(\theta(v_1, v_1, v_3) \rightarrow \beta(v_3))$ (*i.e.*, formula 1).

Trick #2: Replace v_1 in formula (1) with S^q0; the result is a

$$\sigma = \forall v_3(\theta(S^q0, S^q0, v_3) \rightarrow \beta(v_3)). \tag{2}$$

> **Note:** Replacing v_1 with q is more self reference. Sentence σ asserts (in $\mathfrak{N}_E$) that $\#(\sigma)$ is in the set defined by β.

Finally, for this σ we shall prove our original claim that

$$\mathbb{A}_E \models (\sigma \leftrightarrow \beta(S^{\#(\sigma)}0)). \tag{3}$$

Recall that, since q is in fact a natural number, in every model of $\mathbb{A}_E$ there is a unique element x which satisfies $\theta(S^q(0), S^q(0), x)$ — the actual Gödel-number of the substituted formula. How do we create that formula? By definition of θ:

$$\begin{array}{rl} \text{take the formula with Gödel-number } q\text{:} & \forall v_3(\theta(v_1, v_2, v_3) \rightarrow \beta(v_3)) \\ \text{and substitute } S^q0 \text{ for } v_1\text{:} & \forall v_3(\theta(S^q0, S^q0, v_3) \rightarrow \beta(v_3)) \end{array}$$

That formula is σ itself — the substitution operation described by the formula is just the operation we made in defining σ itself! So the Gödel-number of the resultant formula is $\#(\sigma)$. Thus the only element v_3 which can satisfy $\theta(S^q, S^q, 0)$ is $\#(\sigma)$ itself, or, more precisely,

$$\mathbb{A}_E \models \forall v_3(\theta(S^q0, S^q0, v_3) \ \leftrightarrow \ v_3 = S^{\#(\sigma)}0). \tag{4}$$

Reminders:

- σ is the formula $\forall v_3(\theta(S^q0, S^q0, v_3)\rightarrow\beta(v_3))$.
- Assertion (3) was $\mathbb{A}_E \models (\sigma\leftrightarrow\beta(S^{\#(\sigma)}0))$.
- Assertion (4) was $\mathbb{A}_E \models \forall v_3(\theta(S^q0, S^q0, v_3)\leftrightarrow v_3{=}S^{\#(\sigma)}0)$.

Now we prove assertion (3):

$\Rightarrow$: Trivially, from the statement of σ,

$$\sigma \models \theta(S^q0, S^q0, S^{\#(\sigma)}0)\rightarrow\beta(S^{\#(\sigma)}0).$$

By statement 4, $\mathbb{A}_E \models \theta(S^q0, S^q0, S^{\#(\sigma)}0)$.
Thus $\mathbb{A}_E \cup \{\sigma\} \models \beta(S^{\#(\sigma)}0)$.

$\Leftarrow$: Statement (4) implies that

$$\beta(S^{\#(\sigma)}0)\rightarrow[\forall v_3(\theta(S^q0, S^q0, v_3)\rightarrow\beta(v_3))].$$

The part in the square brackets is just σ.

3. **Tarski's Theorem on the Undefinability of Truth:** (for $\mathfrak{N}_E$):
The set $\#(Th(\mathfrak{N}_E))$ is not definable in $\mathfrak{N}_E$.

Proof: Suppose it were definable, say by some formula $\beta(x)$. By the fixed point theorem, *applied to* $\neg\beta$, there is a sentence σ such that

$$\mathfrak{N}_E \models (\sigma\leftrightarrow\neg\beta(S^{\#(\sigma)}0)).$$

But then σ is true in $\mathfrak{N}_E$ iff $\mathfrak{N}_E \models \neg\beta(S^{\#(\sigma)}0)$, contradicting the assumption about β.

Historical Note: This is just a version of the liar's paradox. If β did in fact define the set of statements true in $\mathfrak{N}_E$, σ would "say" "I am false."

4. **Corollary:** *$\#(Th(\mathfrak{N}_E))$ is not r.e.*

Proof: Every r.e. set of (Gödel-numbers of) sentences is Σ_1 definable over $\mathfrak{N}_E$, and Tarski's theorem says that the $\#(Th(\mathfrak{N}_E))$ is not definable over $\mathfrak{N}_E$ by *any* first order sentence.

5. **Corollary: Gödel's Incompleteness Theorem [1931]:**

   If $A \subseteq Th(\mathfrak{N}_E)$ and A is recursive, then $Cn(A)$ is not complete.

   Proof: Since $A \subseteq Th(\mathfrak{N}_E)$, if A were complete, $Cn(A) = Th(\mathfrak{N}_E)$. On the other hand, since A is recursive, $\#(Cn(A))$ is Σ_1 definable on $\mathfrak{N}_E$. This would say that $\#(Th(\mathfrak{N}_E))$ would be first order definable over $\mathfrak{N}_E$, contradiction Tarski's Theorem.

   Corollary: $Th(\mathfrak{N}_E) \neq Cn(\mathbb{PA}_E)$.

   Historical Note: Tracing through the proofs of the fixed-point theorem and Tarski's undefinability theorem, we can mechanically construct a sentence which is true in $\mathfrak{N}_E$ but not provable from $\mathbb{PA}_E$ (given our underlying assumption that $\mathfrak{N}_E \models \mathbb{PA}_E$).

6. **Quotation from Enderton:**

   We can extract more information from the proof of Gödel's theorem. Suppose we have a particular recursive $A \subseteq Th\mathfrak{N}_E$ in mind. Then ... we can find a formula β which defines $\#Cn(A)$ in $\mathfrak{N}_E$. The sentence σ produced by the proof to Tarski's Theorem is (as we noted there) a true sentence *not* in $Cn(A)$.

   This sentence asserts that $\#(\sigma)$ does not belong to the set defined by β, i.e., it indirectly says "I am not a theorem of A." Thus $A \not\vdash \sigma$, and of course $A \not\vdash \neg\sigma$ as well. This way of viewing the proof is closer to Gödel's original proof, which did not involve a detour through Tarski's theorem. For that matter, Gödel's statement of the theorem did not involve $Th(\mathfrak{N}_E)$; we have taken some liberties in the labeling of theorems.

7. **A lemma which says** (roughly) that one can add one new axiom (and hence finitely many new axioms) to a recursive theory without losing the property of recursiveness:

Lemma:

> *If* $Cn(\Sigma)$ *is recursive then* $Cn(\Sigma \cup \{\gamma\})$ *is recursive for any sentence* γ.

Proof: $\Sigma \cup \{\gamma\} \models \alpha$ *iff* $\Sigma \models \gamma \rightarrow \alpha$. Thus $\#(\alpha) \in Cn(\Sigma \cup \{\gamma\})$ iff $\#(\gamma \rightarrow \alpha) \in Cn(\Sigma)$. Since the function mapping each $\#\alpha$ to $\#(\gamma \rightarrow \alpha)$ is recursive (by Theorem concatenation).

8. **Thrm:** **[Strong undecidability of $Cn(\mathbb{A}_E)$]**

Let T be a theory such that $T \cup \mathbb{A}_E$ is consistent. Then $\#(T)$ is not recursive.

Proof: Suppose it were. Since $\#(T)$ is recursive, and $\mathbb{A}_E$ is finite, $\#(Cn(T \cup \mathbb{A}_E))$ is also recursive (by Lemma 7). So it's complement, $\#(\overline{Cn(T \cup \mathbb{A}_E)})$ is also recursive and thus is represented in $Cn(\mathbb{A}_E)$ by some formula β.

By the Fixed Point Theorem, there is a sentence σ such that

$$\mathbb{A}_E \vdash [\sigma \leftrightarrow \neg\beta(S^{\#(\sigma)}0)]. \tag{5}$$

Moreover, since β represents $\#(\overline{Cn(T \cup \mathbb{A}_E)})$, we have that

$$\text{either} \quad \mathbb{A}_E \vdash \neg\beta(S^{\#(\sigma)}0) \quad \text{or} \quad \mathbb{A}_E \vdash \beta(S^{\#(\sigma)}0).$$

From the above 2 lines we also derive that

$$\text{either } \mathbb{A}_E \vdash \sigma \text{ or } \mathbb{A}_E \vdash \neg\sigma.$$

Case 1: $\sigma \in Cn(T \cup \mathbb{A}_E)$**:**

So	$\#(\sigma) \in \#Cn(T \cup \mathbb{A}_E)$	(trivially)
so	$\mathbb{A}_E \vdash \beta(S^{\#(\sigma)})$	(by assumption about β)
so	$\mathbb{A}_E \vdash \neg\sigma$	
so	$\neg\sigma \in Cn(T \cup \mathbb{A}_E)$,	
contradicting	the assumption that T is consistent.	

Case 2: $\sigma \notin Cn(T \cup \mathbb{A}_E)$**:**

We get the reverse contradiction. **¡Left to you!**

9. **Church's Theorem**

The set of logically valid sentences of first order logic with equality is not recursive.

(This holds true in any language which is rich enough — by our proof, any language at least as rich as the language of $\mathfrak{N}_E$: thus any language containing at least one constant symbol, at least 4 function symbols, of which 3 are at least binary (i.e., take at least two arguments), and one relation symbol which is at least binary.)