Math Logic, Week#VIII

Moon's Day

Summary: Some important theorems:

1. Enderton's axiom system is sound.
2. Enderton's axiom system is complete.
3. Compactness Theorem for 1st order logic.
   (We'll see in Chapter III that this shows lots of things cannot be "said" in first order logic.)
4. Enumeration Theorems.

¿Any Q U E S T I O N S on all that?

Finish Exercise 2.2.14: Important point here: we've really seen only one way to show that a relation on a structure is *undefinable*: *automorphisms.*

In Chapter III we'll see a bit more — notably, on structure $\mathfrak{N} = (\mathbb{N}; 0, \text{Succ}, <, +, \cdot, \text{Exp})$, which has no non-trivial automorphisms.

Of course, there is an easy theorem. Since $\mathbb{N}$ is infinite, $\mathfrak{N}$ has uncountably many unary relations (and binary relations, and ternary relations, etc.)

But ther are only countably many formulas in the language of $\mathfrak{N}$, so only countably many of those uncountably many relations (so almost none of those uncountably many relations) can be defined in first order logic.

Homework: **Read** §2.6 & skim §2.7. **Problems:** §2.5 ## 4 & 7.

More exciting homework: **§2.6 ## 1, 5, 8**

§2.6:

Theorem 26A: *If a set Σ of sentences of first order logic has arbitrarily large final models, it has an infinite model.*

Proof: For $k \geq 2$, let λ_k be the formula

$$\exists x_1 \ldots \exists x_k \ \Big(\ (\neg x_2 = x_1) \wedge (\neg x_3 = x_1) \wedge (\neg x_3 = x_2) \wedge \cdots \wedge (\neg x_k = x_1) \wedge \cdots \wedge (\neg x_k = x_{k-1}) \Big)$$

If Σ has arbitrarily large models, then every finite subset of

$$\Gamma \ = \ \Sigma \cup \{\lambda_1, \lambda_2, \ldots\}$$

is satisfiable. So Γ is satisfiable. □

Löwenheim-Skolem Theorem I: *For Γ a set of sentences in a countable (finite or countably infinite) language, if Γ has any infinite models, Γ has a countably infinite model.*

Proof:

- Since Γ has infinite models, $\Gamma \cup \{\lambda_1, \lambda_2, \ldots\}$ is satisfiable.
- Every model of $\Gamma \cup \{\lambda_1, \lambda_2, \ldots\}$ is infinite.
- The models we constructed in proving the completeness theorem were all countable. □

Löwenheim-Skolem Theorem II: *For Γ any set of sentences, if Γ has any infinite models, Γ has infinite models of all cardinalities greater than the cardinality of Γ.*

Proof requires the Axiom of Choice. We won't do it here.

Some vocabulary:

- (Recall that) a *sentence* is a formula with no free variables.

  ¿And why did we pay such attention to bound variables?

- A *theory* is a set Σ of sentences where, whenever $\Sigma \models \alpha$ (for α a sentence in the language of Σ), $\alpha \in \Sigma$.

- A theory Σ is complete (*different meaning* of "*complete*") if, for every sentence α of the language of Σ,

$$\alpha \in \Sigma \quad \text{or} \quad (\neg\alpha) \in \Sigma\,.$$

- For κ an infinite cardinal
  (*e.g.*, $\aleph_0$, the cardinality of the natural numbers, or $2^{\aleph_0}$, the cardinality of the real numbers)
  a theory Σ is κ-*categorical* if all models of Σ of cardinality κ are isomorphic.

Łoś-Vaught Test: *If a theory T in a countable language (i) has only finite models, and (ii) is categorical in some infinite cardinality, then T is complete.*

Example you'll see in book: The theory of dense (strict) linear orderings without endpoints:

$$\begin{aligned}
\forall x \forall y &\;\; (\; x < y \;\vee\; x = y \; y < x \;) \\
\forall x \forall y &\;\; (\; x < y \;\rightarrow\; (\neg y < x) \;) \\
\forall x \forall y &\;\; (\; x < y \;\rightarrow\; \exists z (x < z \wedge z < y \;) \\
\forall x &\;\; (\; \exists y (y < x) \;\wedge\; \exists z (x < z) \;)
\end{aligned}$$

Some models: $(\mathbb{Q}; <)$ ($\mathbb{Q}$ is rational nmbrs.), $(\mathbb{R}; <)$, $((0,1); <)$

Book proves Cantor's theorem that all countable models are isomorphic to each other.

The proof technique, back-and-forth construction, is interesting.

Frea's Day

Change of plan: (because I'm upset people don't know definitions):

> **We shall have an in-class, closed-book, closed-note, (*etc.*), midterm test Friday, March 10**.

Questions shall consist of:

1. **your names**, student numbers, and signatures;
2. **definitions of terms:** quoted, or paraphrased well enough (in *my judgement*);
3. **statements of theorems** – only named theorems;
4. **solutions for some exercises** most of you have claimed.

Next topics: Theories of arithemetic.

(If we can get through that fast enough, I hope to have a couple of weeks to introduce some temporal logics used in formal verification.)

We'll consider 2 or 3 different structues of arithmetic:

- $\mathfrak{N}_S = (\mathbb{N}; 0, S)$, where S is the successor function.
- maybe $\mathfrak{N}_L = (\mathbb{N}; 0, S, <)$.
- $\mathfrak{N}_E = (\mathbb{N}; 0, S, <, +, \cdot, E)$, where E is the exponentiation function.

Extra techniques & topics covered therein:

- Studying a set of axioms by looking at all of its models — and another application of the Łoś-Vaught Test
- *Elimination of quantifiers*: a tool for studying "simple" theories.
- And, in separate notes I'll supply for studying $\mathfrak{N}_E$, classifying sets by the syntactic form of sentences defining them.
- Gödel's first Incompleteness Theorem.