Math Logic, Week#VIII

Moon's Day

Nasty Reminder: Midterm test on Friday; see last week's notes.

- Material **through §2.5** — exercises through §2.4.
- Nasty rule: The test last 50 minutes, so tests turned in after 51 minutes will *lose* credit.
- Always, if you don't understand a question, ask me to explain it. Sometimes I'll explain; sometimes I'll refuse; but I won't penalize you for asking.

  But please ask your questions quietly so no one else will figure out something important from your questions.

As usual: ¿Any questions?

Structure $\mathfrak{N}_S$ and axiom set $\mathbb{A}_S$:

$\mathfrak{N}_S = (\mathbb{N}; 0, S)$, where S is the successor function.

$$\mathbb{A}_S = \left\{ \begin{array}{ll} S1. & \forall x(S(x) \neq 0) \\ S2. & \forall x \forall y(S(x) = S(y) \ \rightarrow \ x = y) \\ S3. & \forall y(y \neq 0 \ \rightarrow \ \exists x(y = S(x))) \\ S4.1. & \forall x(S(x) \neq x) \\ S4.2. & \forall x(S(S(x)) \neq x) \\ \vdots & \vdots \\ S4.n & \forall x(S^n(x) \neq x) \\ \vdots & \vdots \end{array} \right\}$$

Proposition: $\mathfrak{N}_S \models \mathbb{A}_S$ — thus $Cn(\mathbb{A}_S) \subseteq Th(\mathfrak{N}_S)$.

Question: ¿Is $Cn(\mathbb{A}_S) = Th(\mathfrak{N}_S)$?

(If they're equal, then $\mathbb{A}_S$ axiomatizes $Th(\mathfrak{N}_S)$. Otherwise, we wonder what $\mathbb{A}_S$ is missing.)

Another Question: ¿Is $\mathfrak{N}_E$ the only model of A_E?

Answer: ¡No! Consider a theory in language $\{0, S, c\}$ (plus =):

$$\begin{aligned} T_1 &= \mathbb{A}_S \\ &\cup \{c \neq 0,\ c \neq S(0),\ c \neq S(S(0)),\ c \neq S(S(S(0))),\ \ldots\} \end{aligned}$$

¿Is T_1 satisfiable? Let $\mathfrak{A}_1$ be a model of T_1,

(Technically, $\mathfrak{A}_1$ is a structure for the wrong langue, so we'd use it's ***reduct*** to language $\{0, S\}$ — but I want to keep track of the element $c^{\mathfrak{A}_1}$.)

1. ¿What will the interpretations in $\mathfrak{A}_1$ of $0, S(0), S(S(0)), \ldots$ "look like"?
2. ¿ What of c, $S(c)$, $S(S(c))$, …, $S^{-1}(c)$, $S^{-1}(S^{-1}(c))$, … "look like"?
3. ¿Is that all there is in $|\mathfrak{A}_1|$ — the elements we get from 0 and the ones we get from c?

   We don't know; we'd have to go back and further clarify the construction for the completeness theorem.

   ¿But what if we took the ***submodel*** consisting of just those elements and the S relation on them?

¿Can we get yet another model?

¿Can you say anything about what all models "look like"?

Elimination of Quantifiers:

1. *A theory — or set of sentences — T admits elimination of quantifiers iff for every formula φ* (of the language of T) *there is a quantifier-free formula ψ* (of the language of T) *such that*

$$T \models (\varphi \leftrightarrow \psi.$$

   (And, of course, we are interested where we have an algorithm that, given φ, produces ψ.)

2. **Lemma:** *Suppose that, whenever each of α_0, α_1, ..., α_n is an atomic formula or a negated atomic formula, there is a quantifier-free formula ψ where*

$$T \models (\psi \leftrightarrow \exists x(\alpha_0 \wedge \alpha_1 \wedge \cdots \wedge \alpha_n).$$

   Then T admits elimination of quantifiers.

   Proof: By induction on formulas: [and just expanded from Enderton]

   We note that every formula is logically equivalent to one whose only connectives are $\wedge$, $\vee$, and $\neg$, and whose only quantifier is $\exists$.

   If φ is atomic: Trivial: $T \models (\varphi \leftrightarrow \varphi)$.

   If φ is $(\neg\theta)$ where the result holds for θ: Also trivial. **(¿Why?)**

   If φ is $(\theta \vee \chi)$ or $(\theta \wedge \chi)$ where the result holds for θ and χ:
   Similarly trivial.

If φ is $\exists x\theta$ where the result holds for θ:

- By inductive hypothesis, θ is equivalent to a quantifier-free formula.

- And that formula can be put into DNF (disjunctive normal form) — say

  $$T \models \theta \leftrightarrow ((\alpha_1 \wedge \cdots \wedge \alpha_m) \vee (\beta_1 \wedge \cdots \wedge \beta_n) \vee \cdots \vee (\xi_1 \wedge \cdots \wedge \xi_t)).$$

- So, from T we can prove

  $$\exists x\theta \ \leftrightarrow \ \exists x((\alpha_1 \wedge \cdots \wedge \alpha_m) \vee \cdots \vee (\xi_1 \wedge \cdots \wedge \xi_t))$$

  and so also

  $$\exists x\theta \ \leftrightarrow \ \exists x(\alpha_1 \wedge \cdots \wedge \alpha_m) \vee \cdots \vee \exists x(\xi_1 \wedge \cdots \wedge \xi_t)$$

  (¿Why?)

- Now θ was, by assumption, quantifier-free, and the conversion to DNF doesn't add quantifiers.
  So, by definition of DNF (in the context of first-order formulas), each of $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n, \xi_1, \ldots \xi_t$ is either atomic or negated atomic.

- Accordingly, by assumption of the lemma, each of $\exists x(\alpha_1 \wedge \cdots \wedge \alpha_m), \ldots, \exists x(\xi_1 \wedge \cdots \wedge \xi_t)$ is equivalent to a quantifier-free formula.

  And $\exists x\theta$ is equivalent to the disjunction of those quantifier-free formulas, which is still quantifier-free.

3. **Theorem:** *$Th(\mathfrak{N}_S)$ admits elimination of quantifiers.*

Proof: Use the lemma. Consider a formula $\exists x(\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_m)$, where each α_i is atomic or negated atomic.

- Note that each α_i is of the form $t_1 = t_2$ or $(\neg t_1 = t_2)$, for t_1 and t_2 terms. **(¿Why?)**
- And each term is of one of 3 forms: $S^n(x)$, $S^n(0)$, and $S^n(v_i)$, for $v_i \neq x$ — for some natural number n for each term.

Enderton writes out the algorithm — so I'll just do some examples that show the ideas.

(a) θ is $\underline{v_j \neq S^{13}(v_i) \;\wedge\; S^3(v_i) = S^7(v_k) \;\wedge\; S^9(0) = S(v_j) \;\wedge\; S^3(0) \neq S^2(v_i)}$

x does not appear in θ, so the $\exists x$ is redundant
— and $\exists x\theta$ is logically equivalent to θ.

(b) θ is $\underline{x \neq S^{13}(v_i) \;\wedge\; \mathbf{S^3(x) = S^7(0)} \;\wedge\; S^9(x) = S(v_j) \;\wedge\; S^3(0) \neq S^2(x)}$

The formula *requires that* $S^3(x) = S^7(0)$, so x *must be* $S^4(0)$.
— so $\exists x\theta$ is equvalent in $Th(\mathfrak{N}_S)$ to

$S^4(0) \neq S^{13}(v_i) \;\wedge\; \mathbf{S^7(0) = S^7(0)} \;\wedge\; S^{13}(0) = S(v_j) \;\wedge\; S^3(0) \neq S^6(0)$

(just replacing each occurrence of x with $S^4(0)$ and dropping the now redundant quantifier).

(c) θ is $\underline{x \neq S^{13}(v_i) \;\wedge\; \mathbf{S^7(x) = S^3(0)} \;\wedge\; S^9(x) = S(x) \;\wedge\; S^3(0) \neq S^2(x)}$

Note that $S^7(x) = S^3(x)$ is always false in $\mathfrak{N}_S$.
— so $\exists x\theta$ is equivalent (in $Th(\mathfrak{N}_S)$) to $0 \neq 0$.

(d) $\theta = \underline{x \neq S^{13}(v_i) \;\wedge\; \mathbf{S^3(x) = S^7(v_k)} \;\wedge\; S^9(x) = S(v_j) \;\wedge\; S^3(0) \neq S^2(x)}$

Same idea part (3b): replace x with $S^4(v_k)$ and drop the $\exists x$.

(e) θ = $\underline{x \neq S^{13}(v_i) \ \wedge \ \mathbf{S^7(x) = S^3(v_i)} \ \wedge \ S^9(x) = S(v_j) \ \wedge \ S^3(0) \neq S^2(x)}$

This says $S^4(x) = v_i$, so $x = S^{-4}(v_i)$

— but we have no symbol in the language for S^{-1}.

Furthermore, we don't know of $S^{-4}(v_i)$ even exists.

But *if it does exist*: since S is 1-1 in $\mathfrak{N}_S$: in $\mathfrak{N}_S$,

$S^3(S^{-4}(v_i) = S^k(y)$ is equivalent to $S^3(v_i) = S^{k+4}(y)$.

So $\exists x\theta$ is equivalent in $Th(\mathfrak{N}_S)$ to

$$\mathbf{v_i \neq 0 \wedge v_i \neq S(0) \wedge v_i \neq S(S(0)) \wedge v_i \neq S(S(S(0)))}$$

$$\wedge$$

$$v_i \neq S^{17}(v_i)) \ \wedge \ \mathbf{S^7(v_i) = S^7(v_i)} \ \wedge \ S^9(v_i) = S^5(v_j) \ \wedge \ S^6(0) \neq S^2(v_i)$$

(f) θ is $\underline{\mathbf{x \neq S^{13}(v_i)} \ \wedge \ \mathbf{S^7(x) \neq S^3(v_i)} \ \wedge \ S^9(0) = S(v_j) \ \wedge \ S^3(v_j) \neq S^2(v_i)}$

Here, all the references to x are negative

— so (for any fixed interpretation s), all those references say about x is that it is not one of finitely many elements

— and $|\mathfrak{N}_S|$ is infinite, so there always is such an x.

So $\exists x\theta$ is equivalent, in $Th(\mathfrak{N}_S)$, to

$$\mathbf{0 = 0} \ \wedge \ \mathbf{0 = 0} \ \wedge \ S^9(0) = S(v_j) \ \wedge \ S^3(v_j) \neq S^2(v_i).$$

4. Enderton notes that the above theorem can be expanded to another proof that $Cn(\mathbb{A}_S) = Th(\mathfrak{N}_S)$:

   (a) Adjust proof to show elimination of quantifiers in $Cn(\mathbb{A}_S)$,

   (b) Show that every *quantifer-free sentence* is provably true or provably false from $\mathbb{A}_S$,

   (c) And, again, use the fact that, since $\mathfrak{N}_S \models \mathbb{A}_S$, $Cn(\mathbb{A}_S) \subseteq Th(\mathfrak{N}_S)$.