Points and Lines in the Plane

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Abstract

This thesis will focus on two topics: (1) finding intersections determined by an arrangement of hyperplanes (e.g., lines in a plane), and (2) lower bounds on the number of various "types" of lines determined by a configuration of points.

The first topic is algorthmic. Given an arrangement of n lines in $\mathbb{R}^2$, a $O(n \log n)$ algorithm is demonstrated for finding an ordinary intersection (i.e., an intersection of exactly two lines). This algorithm is then extended to finding an ordinary intersection among hyperplanes in $\mathbb{R}^d$, under the hypothesis that no d hyperplanes pass through a line and not all pass through the same point. Algorithms are also given to find an ordinary intersection in an arrangement of pseudolines in time $O(n^2)$, and to find a monochromatic intersection in a bichromatic arrangement of pseudolines in time $O(n^2)$.

The second topic is combinatorial. Let G and R be finite sets of points, colored green and red respectively, such that $|G| = n$, $|R| = n - k$, $G \cap R = \emptyset$, and $G \cup R$ are not all collinear. Lower bounds will be demonstrated for several types of lines (e.g., bichromatic and equichromatic) determined by few points in $\mathbb{R}^2$.

Dedicated to my darling wife, Katrina.

Acknowledgements

First, I would like to thank my advisor, Prof. George B. Purdy. He directed my research toward problems that were both interesting and solvable. I am grateful for his patience and persistence with me as his student. Many of the results of this thesis were a culmination of our joint efforts, and several of the results may be solely attributed to him. (I will attempt to mark those results as such.) His direction has led me to areas of research that I hope to study throughout my academic career.

I also thank the faculty and staff at the University of Cincinati. They have created an excellent environment for learning and research.

Most of all, I thank my wife Katrina. She sacrificed much so that I could leave a profitable career and return to graduate school. Her hard work, patience, and love, I will always admire. This thesis is dedicated to her.

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Chapter 1

Introduction

Many (if not most) problems in discrete mathematics can be intuitively understood by a well motivated person. Some of these problems even have a semblance to the "brain teasers" one might see published in a newspaper. Although they seem intuitive, their solutions (if known) are often quite complicated.

Among the most intriguing problems in discrete mathematics are those with a geometric aspect. When a problem is stated geometrically, one might feel that intuition provides a significant advantage in finding the solution. However tantalizing these may be to researchers, many of these "intuitive" problems are still open, i.e., no satisfactory solution is known. These attributes make such problems captivating for study. Several books are devoted to discussing open problems in combinatorial (or discrete) geometry (e.g., [3], [4] or [5]) with the hope that a reader will be inspired to find new insights or, perhaps, even find a solution.

The present thesis presents several results from the fields of combinatorial and computational geometry. These fields each contain interesting questions about the nature of geometrical structures. As the name suggests, combinatorial geometry studies discrete aspects of geometry, often attempting to answer questions about the enumeration (or counting) of geometric objects. Similarly, computational geometry is a search for the most efficient methods to perform computation on geometric objects. These two fields often have a symbiotic relationship, i.e., results in one field affect the other.

Combinatorial geometry, like several other fields of mathematics, has Paul Erdős as a protagonist. It was a question published by him in 1943 ([6]) about points and the lines that instigated the study of such discrete structures. Subsequent to his question, other famous mathematicians (e.g., Th. Motzkin, G. A. Dirac, and L. M. Kelly) followed his lead into this subject. Erdős' questions not only instigated, but also were a driving force behind the development of this field.

A mention should also be made of Michael I. Shamos whose 1978 thesis ([7]) was the first to address computational questions about geometry. After his thesis, the field of computational geometry developed quickly. Within a few years, the first significant book on the subject was published, [8], for which Shamos is a co-author.

I hope the reader would agree that only minimal background knowledge is necessary to understand and appreciate the results recorded here. For readers who are beginning their study of these topics, Chapter 2 will provide some of the necessary context from which to begin study.

Chapters 3 and 4 contain the primary results for this thesis. Both of these chapters have been submitted for publication, and Chapter 4 has already appeared in [9].

Chapter 5 concludes with a brief discussion of possible future work to extend these results. Results were found, under the direction of my adviser, between Spring of 2008 and Summer of 2009.

Chapter 2

Background

2.1 A Question Of J.J. Sylvester

In 1893, J.J. Sylvester posed the following problem [10],

> Prove that it is not possible to arrange any finite number of real points so that a right line through every two of them shall pass through a third, unless they all lie in the same right line.

A simpler way might be to ask, "Does every set of noncollinear points necessarily determine an ordinary line (i.e. a line passing through exactly two of them)?".

Sylvester's problem remained without a solution (and was perhaps even forgotten) until it was raised again independently by Erdős in the 1930s. In 1943, Erdős published this question in *American Mathematical Monthly* [6], and a solution appeared soon after in [11]. The first solution to this problem is commonly attributed to T. Gallai, thus it is now called the Sylvester-Gallai Theorem.[1]

Although several proofs of the Sylvester-Gallai Theorem are known, one credited to L.M. Kelly (and published by Coxeter in [13]) is considered the most elegant. His proof is as follows.

Proof. Assume we have a noncollinear configuration of points, $\mathcal{P}$, such that any

[1] Erdős was sure that Sylvester had a proof of his conjecture, but it (apparently) was never published.[12]

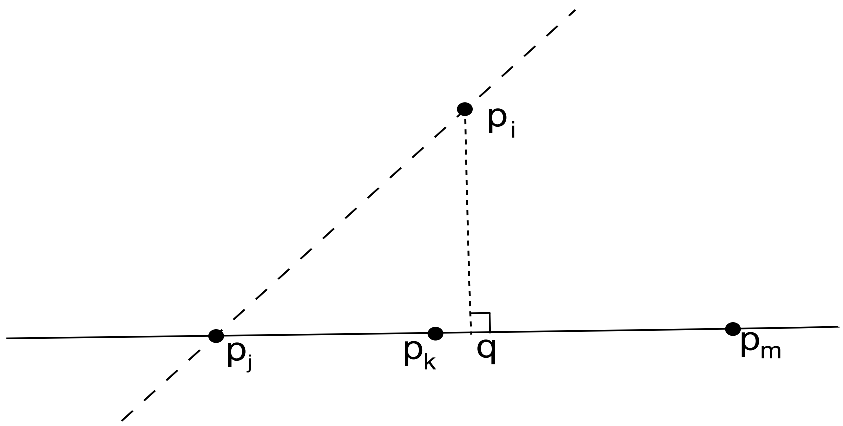

Figure 2.1: A perpendicular from point p_k to line p_ip_j is shorter than the perpendicular from p_i to line p_jp_k.

line determined by two points also passes through a third. Assume the points of $\mathcal{P}$ are numbered $p_1, p_2, \ldots, p_n$. There exists a point, p_i, and a line, p_jp_k, such that the point-line pair (p_i, p_jp_k) determine the smallest nonzero distance along a perpendicular among all point-line pairs of the configuration. Let q be the foot of the perpendicular from p_i to p_jp_k. Let p_m be a third point point on line p_jp_k. (See Figure 2.1.) Among the three (or more) points on the line, i.e. p_j, p_k, and p_m, two must lie to one side of q. Assume they are ordered on the line $p_jp_kqp_m$ (q might coincide with p_k). The point-line pair (p_k, p_ip_j) forms a shorter perpendicular distance, and thus a contradiction. $\square$

2.2 Duality and the Projective Plane

Consider the similar problem, "Does every arrangement of lines, no two being parallel and not all passing through a common point, necessarily determine an ordinary intersection (i.e. an intersection of exactly two lines)?". The answer is "yes". In fact, this problem is equivalent to that of Sylvester. The equivalence of these two questions is due to a *duality* that exists between points and lines in a projective plane.

Euclid's *Elements* describes plane geometry in terms of constructions (e.g., triangles, squares or etc.) attainable via compass and straight-edge. Essential to many of the constructions is the concept of a right angle. Consider a line l_0 and two lines which cross it, l_1 and l_2. By the last of Euclid's five postulates, called the "parallel postulate", these crossing lines, l_1 and l_2, will intersect on whichever side of l_0 having interior angles summing to less than two right angles, i.e., $180°$. If l_1 and l_2 intersect at all, one of the two pairs of interior angles must sum to less than $180°$ by Proposition 17 of Book One in Euclid's *Elements*. Thus, by constructing right angles at two distinct points on a line, one creates two parallel lines, i.e., two lines which do not intersect. However, to obtain a "duality" between points and lines, parallel lines cannot be allowed.

Although Euclid's five postulates are quite natural, perhaps to the point of seeming essential, one can also construct interesting alternative geometries using less restrictive sets of postulates (or equivalently, axioms). Considered most general among the "alternative" geometries is Projective Geometry. In this geometry, there is no concept of measure, of angles or distance, or "betweeness", among three points on a line. In contrast to Euclidean, Projective Geometry asks "How much remains if we discard the compass . . . , and use the straight-edge alone?" [14, p. 1].

A common conceptual model given for the projective plane (i.e., two-dimensional Projective Geometry) is that of a sphere with antipodal points considered equivalent. (See Felsner's description [15, pp 70-72].) With this model, one can think of a plane, π, (on which we have, e.g, points and lines) as lying tangent to this sphere. Thus, lines in the plane π correspond (via *projection* from the sphere's center) to great circles on the sphere. Furthermore, there exists a unique great circle that lies in a plane parallel to π, and, thus, does not project onto π (at least not at any set of finite coordinates). This unique great circle corresponds to the so-called *line at infinity*.

With some inspection, one can verify that lines which are parallel on the plane correspond to great circles on the spehere that intersect at the line at infinity. Interestingly, for an arrangement of lines in the plane, one may designate one (or none) of them as the line at infinity. Thus, the same arrangement of lines in the plane may be projected in several different ways. So equivalence between two arrangements of lines is determined by the incidence relationships within each arrangement, and not the angles or distances between lines. It is *incidence* that is of primary concern.

Because incidence defines a unique arrangement in the projective plane, there exists a "Principle of Duality" by which one may interchange points and lines while preserving the analogous incidence relationships. Such an interchange, is called a "duality mapping". (Several such mappings are discussed in the subsequent subsections.) Of course, one must also exchange other concepts for their analog, e.g., "concurrent lines" for "collinear points". For example, from the claim "any two points determine a connecting line," one may derive the claim "any two lines determine an intersection point."

In Chapter 3, an algorithm is demonstrated that finds an ordinary intersection (i.e., an intersection involving exactly two lines) among an arrangement of lines. It is claimed in that chapter that, by using a duality transform, the algorithm can be utilized to find an ordinary line among an arrangement of points. Since the Principle of Duality does not hold for the Euclidean plane (because incidence is not the only concern in such a context), one might wonder how this concept might be used to reduce an algorithmic problem concerning points to the analogous problem concerning lines. In order for our claims to be made evident, we must further describe how to implement a duality mapping (or transform).

A *duality transform* is a function which provides a mapping between points and lines (in $\mathbb{R}^2$) such that their incidence relationships are preserved. For example, let $\mathbb{D}$ be a duality tranform. For any three collinear points p_1, p_2, and p_3, the lines $\mathbb{D}(p_1)$, $\mathbb{D}(p_2)$ and $\mathbb{D}(p_3)$ must be concurrent (i.e., intersect at a common point). Conversely, any three concurrent lines should transform into three collinear points.

Again, let $\mathbb{D}$ be a duality transform. The following two properties are assumed (as axioms). In the subsections below, specific duality transforms will be shown to have these two properties.

Axiom 2.2.1. $\mathbb{D} = \mathbb{D}^{-1}$*, i.e.* $\mathbb{D}$ *is its own inverse.*

Axiom 2.2.2. *Point* $\mathbb{D}(L)$ *lies on line* $\mathbb{D}(p)$ *iff point* p *lies on line* L*.*

Following from these two axioms is the following theorem:

Theorem 2.2.3. *Points* p_1 *and* p_2 *determine line* L *if and only if lines* $\mathbb{D}(p_1)$ *and* $\mathbb{D}(p_2)$ *intersect at point* $\mathbb{D}(L)$*.*

Proof. Assume points p_1 and p_2 determine line L, but lines $\mathbb{D}(p_1)$ and $\mathbb{D}(p_2)$ do not intersect at point $\mathbb{D}(L)$. In this case at least one of the following is true:

- Point $\mathbb{D}(L)$ does not lie on $\mathbb{D}(p_1)$.
- Point $\mathbb{D}(L)$ does not lie on $\mathbb{D}(p_2)$.

By Axiom 2.2.2, this contradicts our original assumption.

The reverse implication follows from Axiom 2.2.1. □

One should note that the duality transforms discussed here are not bijective functions (i.e., one-to-one correspondences). In fact, such a bijective function cannot exist for $\mathbb{R}^2$. To see why this is true, for sake of contradiction, assume that $\mathbb{D}$ is a bijective duality transform for $\mathbb{R}^2$, and let l_1 and l_2 be two distinct parallel lines in $\mathbb{R}^2$. The two points $\mathbb{D}(l_1)$ and $\mathbb{D}(1_2)$ must be distinct points that, subsequently, determine a line l_3. Since $\mathbb{D}$ must preserve incidence, $\mathbb{D}(l_3)$ should be a point in $\mathbb{R}^2$ incident to both l_1 and l_2, but no such point exists and, thus, a contradiction. The reader will see below how these limitations are overcome in practice.

The next two sections will demonstrate two commonly used duality transforms. See [14, p. 15] for an axiomization of Projective Geometry. For a discussion of duality in the projective plane see [14, pp. 24-32]. See [16, pp. 201-205] and [17, pp. 12-15] for further discussion of duality tranforms.

2.2.1 Polar Dual

Perhaps the first duality transform in common use is called the *polar dual*. It maps point (a, b) to line $ax + by = 1$[2] and vice versa.

One can quickly see that this transform is only a partial function of lines to points, since not every line can be expressed by an equation of the form $ax + by = 1$ (i.e the lines which pass through the origin, e.g. $x + y = 0$). The lines, which are not mapped, would conceptually correspond to points on the *line at infinity* and require special treatment. This is consistent with our representation of points by ordered pairs of numbers, since points on the line at infinity also cannot be represented by a pair of real numbers. In practice, a simple translation would be used so that no line in an arrangement passes through the origin.

From the definition, it is apparent that this mapping possesses the property required by Axiom 2.2.1. The following is proof that it also has the property required by Axiom 2.2.2.

Proof. Let P be the point at (a, b) which lies on a line L given by the equation $cx + dy = 1$. Since P lies on L, we know that $ca + db = 1$.

By definition, $\mathbb{D}(P)$ is the line given by equation $ax + by = 1$, and $\mathbb{D}(L)$ is the point at (c, d). Obviously $ca + db = ac + bd = 1$, thus, we can see that point $\mathbb{D}(L)$ lies on line $\mathbb{D}(P)$.

One can see that the converse is also true, completing the proof. □

2.2.2 Parabolic Dual

Within Computational Geometry, a more common duality transformation is the "parabolic dual" [3]. It maps the point at (a, b) to the line given by equation $y = 2ax - b$ and vice versa.

Just as with the polar dual, this is only a partial function since vertical lines

[2] The line $ax + by = -1$ is also commonly used.

[3] The author has not found the term "parabolic dual" in the literature, although it seems to follow naturally from the description given by O'Rourke in [16, p 202].

(e.g. x=2) cannot be represented in the form required. Likewise, these (vertical) lines correspond through duality with points on the line at infinity. Of course for any given (finite) arrangement of lines, vertical lines can be detected. A subsequent rotation can then be used to create a different arrangement without vertical lines that preserves the incidence relationships.

From the definition, it is apparent that this mapping possesses the property required by Axiom 2.2.1. The following is proof that it also has the property required by Axiom 2.2.2.

Proof. Let P be the point at (a, b) which lies on a line L given by the equation $y = 2cx - d$. Since P lies on L, we know that $b = 2ca - d$.

By definition, $\mathbb{D}(P)$ is the line given by equation $y = 2ax - b$, and $\mathbb{D}(L)$ is the point at (c, d). Since we know that $b = 2ca - d$, and thus $d = 2ac - b$, we can see that point $\mathbb{D}(L)$ lies on line $\mathbb{D}(P)$.

One can see that the converse is also true, completing the proof. □

2.3 Melchior's Inequality

Prior to Gallai's proof in [11] (and even before Erdős had published his question in *American Mathematical Monthly*), an inequality was published in *Deutsche Mathematik* that provides proof of the Sylvester-Galai Theorem [18]. Let S be an arrangement of lines. Let t_k be the number of points through which k lines of S intersect. Assuming $|S| \geqslant 3$ and not all lines pass through the same intersection point, the following (i.e., *Melchior's Inequality*) holds:

$$\sum_{k \geqslant 2} (k-3) t_k \leqslant -3$$

This immediately shows that there are at least three "ordinary" intersection points, i.e., by rearranging the terms one can derive $t_2 \geqslant 3 + \sum_{k \geqslant 4} (k-3) t_k$. It further demonstrates that for every point incident to four (or more) lines, there exists at least one other point incident to exactly two.

2.3.1 Euler's Polyhedral Formula

In order to prove Melchior's inequality, one may start from Euler's polyhedral formula. This formula demonstrates an invariant relationship between the vertices, edges, and faces determined by a plane graph. (A *plane* graph is the drawing of a graph in the plane such that each vertex is distinct and no two edges touch. By definition such a graph, capable of being drawn this way, is a *planar* graph.) Let V be the set of vertices, E the edges, and F the faces (or planar regions) determined by the graph. Euler's polyhedral formula says that $|V| - |E| + |F| = 2$. No fewer than 19 proofs exist for this formula [19]. See [20, p 65] and [15, p 4] for two (distinct) simple proofs.

A similar invariant exists for the vertices, edges and faces determined by an arrangement of lines in the projective plane. Any example quickly shows that the analogous formula for an arrangement of lines is instead:

$$|V| - |E| + |F| = 1$$

An informal proof can be seen by returning to the spherical model for the projective plane (see Section 2.2).

Let A be an arrangement of lines. Let S be an arrangement of great circles on a sphere that would (each) project onto the lines of A. We may now construct a graph. Consider the intersection of any two great circles to be a vertex. Two vertices are adjacent if they they are adjacent intersection points on a great circle. (A graph drawn on a sphere is equivalent to one drawn on the plane. This can be seen by "stretching" an arbitrary face of the drawing out onto the plane.) Since we have constructed a plane graph, its constituents are correlated by Euler's formula, $|V| - |E| + |F| = 2$. Now return to our original arrangement of lines. Each intersection of lines corresponds to two intersections (at antipodal points) on the sphere. Likewise, the faces and "edges" are counted twice. Thus, $2(|V| - |E| + |F|) = 2$, or equivalently $|V| - |E| + |F| = 1$.

We now provide a more formal inductive proof (similar to the one provided by Felsner in [15, p 73]).

Lemma 2.3.1. *Given an arrangement of lines in the projective plane, let V be the set of vertices, E the edges, and F the faces determined by the arrangement. The following invariant must hold:*

$$|V| - |E| + |F| = 1$$

Proof. Let S be an arrangement of lines in the projective plane. Given $|S| = 2$, we have $|V| = 1$, $|E| = 2$ and $|F| = 2$ (i.e. edges and faces "wrap around" in the projective plane).

Assume it is true for $|S| = n$. Let l be a line not in S, and let $S' = S \cup \{l\}$ so that $|S'| = n+1$. Let V', E', and F' be the vertices, edges and faces, respectively, of S'. By adding l to S additional vertices, edges, and faces are created. For each edge created by l in E', a face of F is split into two. For each vertex created by l in V', an edge E is split into two. Therefore, the value of $|V| - |E| + |F|$ is not affected by the addition of l, i.e., $|V'| - |E'| + |F'| = |V| - |E| + |F| = 1$. □

2.3.2 Melchior's Proof of the Inequality

One can find proof of Melchior's inequality in the original article, [18]. The article is in German and might not be readily accessible to the reader. Hence, a proof similar to that of Melchior is provided below.

Theorem 2.3.2 (Melchior's Inequality)**.** *Let t_k be the number of intersection points through which k lines pass. Given an arrangement of lines, not all concurrent, in the projective plane, the following inequality holds:*

$$\sum_{k \geqslant 2} (k-3) \cdot t_k \leqslant -3$$

Proof. Let f_k be the number of faces determined by a set of exactly k edges from the arrangement, and let $F = \sum_{k \geqslant 3} f_k$ be the total number of faces. (Note that when an arrangement of lines is not all concurrent each face has at least three sides.) Let t_k be the number of points (i.e., intersections) at which k lines cross. Let E be the number of edges, and $V = \sum_{k \geqslant 2} t_k$ be the number of vertices. We start with the following identity:

$$2 \cdot \sum_{k \geqslant 2} k \cdot t_k = 2E = \sum_{k \geqslant 3} k \cdot f_k$$

By combining this with Lemma 2.3.1 (i.e., $F = \sum_{k \geqslant 3} f_k = E + 1 - V$), we get the inequality:

$$2E = \sum_{k \geqslant 3} k \cdot f_k \geqslant 3F = 3E + 3(1 - V)$$

Thus,

$$2E - 3F = 3(V - 1) - E \geqslant 0$$

From this we can derive Melchior's inequality:

$$\begin{aligned} 3(V-1) - E &= 3V - E - 3 \\ &= 3 \cdot \sum_{k \geqslant 2} t_k - \sum_{k \geqslant 2} k \cdot t_k - 3 = \sum_{k \geqslant 2} (3 - k) \cdot t_k - 3 \geqslant 0 \end{aligned}$$

Therefore,

$$\sum_{k \geqslant 2} (k - 3) \cdot t_k \leqslant -3$$

□

2.4 How Many?

A natural extension to Sylvester's problem is to ask *how many* ordinary lines must be determined by a set of n non-collinear points. Let $t_2(n)$ be the minimum number of ordinary lines determined over all sets of n non-collinear points in the plane. From Kelly's proof (in Section 2.1) we know $t_2(n) \geqslant 1$, and from Melchior's inequality (in Section 2.3) we know $t_2(n) \geqslant 3$. Is it possible that, as n gets larger, $t_2(n)$ never surpasses some constant? Erdős and de Bruin asked a

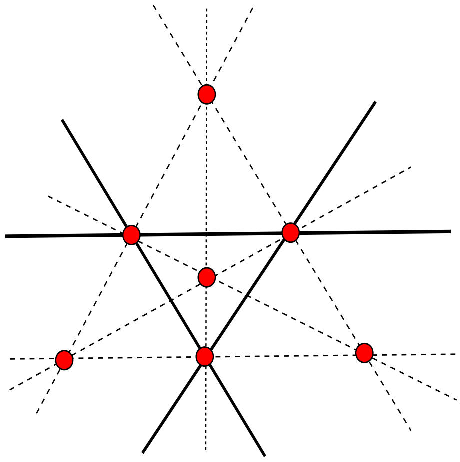

Figure 2.2: The Kelly-Moser configuration has seven points that determine only three ordinary lines, and thus, $t_2(7) = 3$.

similar question in [21]. More specifically, they asked whether $\lim_{n\to\infty} t_2(n) = \infty$. In 1951, Motzkin confirmed this by showing that $t_2(n) \geqslant \sqrt{n}$ ([22]) [4]. The same year, in [24] Dirac conjectured that $t_2(n) \geqslant \lfloor \frac{n}{2} \rfloor$ for all n.

Only two specific cases are known for which n non-collinear points determine less than $\lceil \frac{n}{2} \rceil$ orindary lines. The first is the “Kelly-Moser configuration”, published in [25], of seven points that determine only three ordinary lines. In the same paper, Kelly and Moser prove $t_2(n) \geqslant \frac{3n}{7}$. Their configuration thus demonstrated that “in a certain sense, this is a best possible bound”[25]. See Figure 2.2.

The other known configuration for which $t_2(n) \leqslant \lceil \frac{n}{2} \rceil$, due to McKee and published in [1], consists of thirteen points that determine only six ordinary lines. Eight of the points are on the vertices of two adjacent pentagons, with the midpoint of the shared side containing an additional point. The remaining four points are on the line at infinity. See Figure 2.3.

In 1981 Hansen published, in [26], for his habilitation an erroneous proof that $t_2(n) \geqslant \frac{n}{2}$. Reportedly, his proof was difficult to read and dubious from the time of its publication. Csima and Sawyer made a careful study of Hansen’s proof. They discovered his error, and ultimately improved the bound published

[4] This 1951 paper of T.S. Motzkin introduced the term *ordinary line* (see [23]).

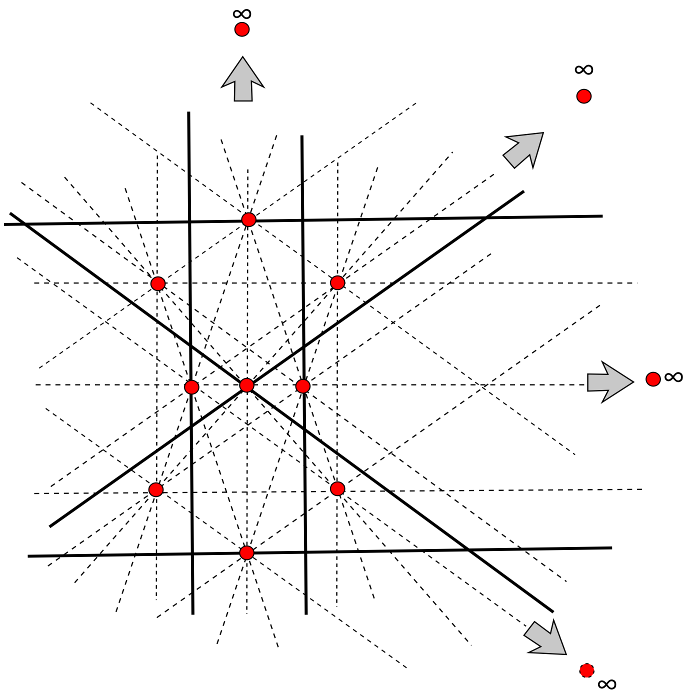

Figure 2.3: McKee's configuration showing thirteen points determining only six ordinary lines.

n	3	4	5	6	7	8	9	10	11	12	13
$t_2(n)$	3	3	4	3	3	4	6	5	6	6	6

Figure 2.4: Known values of $t_2(n)$ originally published in [1].

by Kelly and Moser. Csima and Sawyer demonstrated in [27] that, with the exception of $n = 7$, $t_2(n) \geqslant \frac{6n}{13}$. This bound is again "best possible", in a certain sense, because of McKee's configuraton of thirteen points.[5] For $n \leqslant 13$, all values of $t_2(n)$ are known and were published by Crowe and McKee in [1]. A table of these values is provided in Figure 2.4.

Asymptotically, Dirac's conjectured bound cannot be improved. An infinite family of configurations, due to Károly Böröczky and published in [1], is known which demonstrates $t_2(2m) \leqslant m$ for all m. Consider m points at the vertices of a regular m-gon. The $\binom{m}{2}$ pairs of points determine only m distinct directions. Thus, also select the m points on the line at infinity corresponding to these m

[5]The author finds it interesting that both proven linear bounds on t_2 have a closely related configurations demonstrating the bound to be "best possible".

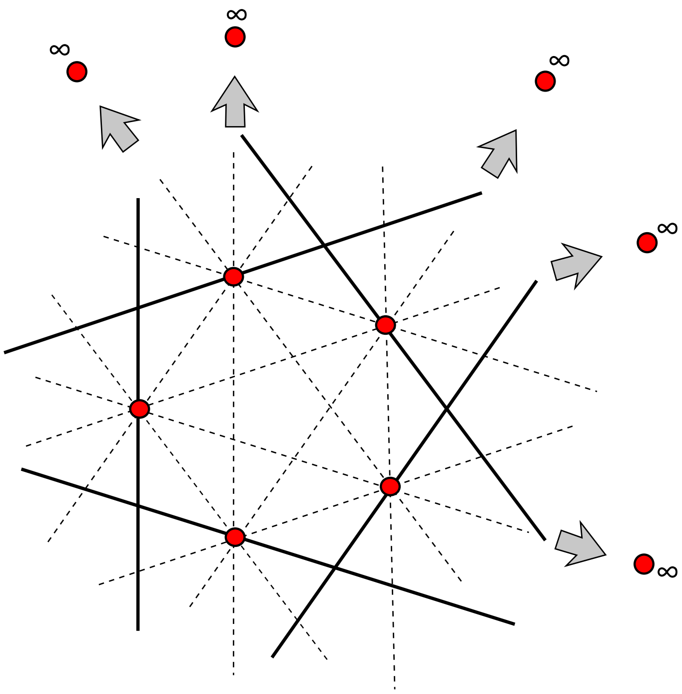

Figure 2.5: A Böröczky configuration of ten points determining five ordinary lines.

directions. Such a configuration determines only m ordinary lines, i.e., the line connecting each vertex, v, to the point on the line at infinity determined by it's neighboring vertices. See Figure 2.5 for the configuration for $m = 5$.

2.5 Da Silva and Fukuda's Conjecture

Let P be a set of n points, and L be a set of m lines in $\mathbb{E}^2$. The set L is said to *isolate* P if every point in P lies in a distinct region of $\mathbb{E}^2 \setminus L$.

In [28] Da Silva and Fukuda study minimal arrangements of lines which isolate a point set, i.e., $min_{(P)}(|L'| \in \{L : L \text{ isolates } P\})$. They demonstrate a method to compute such a minimal arrangement. When the resulting arrangement is allowed to instead consist of "pseudolines" their method runs in time $O(|P| \log(|P|))$. (See Section 3.4 for a brief description of pseudolines.)

In the conclusion of their paper ([28]), they made the following conjecture.

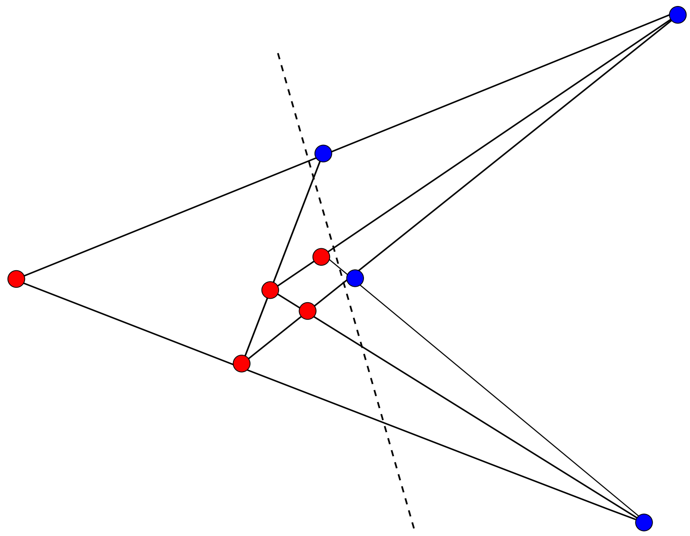

Figure 2.6: A counter example to Da Silva and Fukuda's conjecture. Originally published in [2].

Conjecture 2.5.1. Let S be a set of n point in $\mathbb{R}^2$. Let l be a line that contains no point of S and partitions S into two subset, B and R, such that $S_1 \leqslant S_2 \leqslant \lceil \frac{n}{2} \rceil$. There exist points $p_1 \in B$ and $p_2 \in R$ such that the line connecting them passes through no other point from S.

Unfortunately, this conjecture was shown to be false for $|S| = 9$. Using oriented matroids, Finschi and Fukuda enumerated all possible arrangement of few points in search of a counter-example ([2]). The conjecture holds for point sets containing eight or fewer points. However, for nine points they found a counter-example. See Figure 2.6. Because of the combinatorial explosion (e.g., 15 296 266 possible "abstract order types" for nine points) [6], it would likely be difficult to extend their search much further (beyond nine points). The conjecture is still open for larger point sets.

One might consider the partitioning of the point set by a line, as done by Finschi and Fukuda, as establishing a "coloring" for the point set; That is, one side might be, e.g, blue and the other side red. This leads one to the concept of "bichromatic" point sets. Conjectures concerning bichromatic sets were made

[6]The author does not know whether a bounded growth rate for the number of abstract order types is known.

as early as 1965 when Ron Graham asked whether a bichromatic arrangement of lines (i.e., the dual context) necessarily determines a “monochromatic” intersection point. Chakerian later, in [29], published a proof that a “monochromatic” intersection point is necessarily determined.

Da Silva and Fukuda’s conjectured (stronger) version of the Sylvester-Gallai theorem provided motivation Pach and Pinchasi to further study bichromatic point sets [30]. Their primary results related to how many lines, in proportion to the total, pass through few points (e.g. $\leqslant 6$) and at least one of each color. The work of Pach and Pinchasi, in turn, motivated the author and his advisor to attempt to improve their bounds. These results are covered in Chapter 4.

Chapter 3

On Finding Ordinary Or Monochromatic Intersection Points

3.1 Introduction

Over a century ago Sylvester posed the question of whether a set of n non-collinear points necessarily determines an ordinary line [10]. (An *ordinary* line is one incident to exactly two points.) Although it was thought to be true, no proof was found until the problem was raised again by Erdős in the 1930's. Soon after, it was proven by Gallai and his proof was published in [11]. Hence, it is now called the Sylvester-Gallai Theorem. (See also [13] for an elegant proof by L. M. Kelly.)

Since Sylvester originally posed his question in 1893, a variety of related questions have been asked. One well known variation relates to a two-colored, or bichromatic, set of points. Ron Graham first asked (around 1965, see [31]) whether a bichromatic set of non-collinear points necessarily determines a monochromatic line, i.e., a line determined by two or more points all of which are the same color. The first published proof was a few years later by Chakerian [29]. Earlier than Chakerian (and referenced in his paper), Motzkin and Rabin had proofs of this result in its dual form. (Motzkin's proof was published in [23][1].) This theorem is now commonly called the Motzkin-Rabin Theorem.

The algorithms in this article deal with arrangements of hyperplanes (i.e., $(d-1)$-flats in $\mathbb{R}^d$) or pseudolines in the euclidean plane. By duality, the algorithms on hyperplanes can be used, as well, on a point configuration to solve the dual problem. However, hyperplane arrangements are more general than a dual of points, e.g., any two points determine a line, but two parallel lines do not determine an intersection point. Thus, some problem instances (i.e., those involving parallel hyperplanes) can only be solved by algorithms that work in the domain of hyperplane arrangements.

The first algorithm presented finds an ordinary intersection point in an arrangement of lines (some possibly parallel) in $\mathbb{R}^2$ in time $O(n \log n)$. This algorithm will subsequently be used to solve the same problem for hyperplanes in $\mathbb{R}^d$.

3.2 Ordinary Points in an Arrangement of Lines in $\mathbb{R}^2$

3.2.1 Existence of Ordinary Intersection Points

Dirac conjectured in 1951 that in any set of $2n$ points, there exist n ordinary lines [24]. The best known lower bound is $\frac{6n}{13}$ ordinary lines in a set of n points, found by Csima and Sawyer in [27]. This improved upon the Kelly and Moser result of $\frac{3n}{7}$ [25].

Since an ordinary line always exists among a set of non-collinear points, an obvious question within computational geometry is how to find one. A naive method would potentially take time $O(n^3)$ by considering for each point pair

[1]Grünbaum states in [31] that the proof published in [23] is actually due to Motzkin, although the text attributes it to S. K. Stein.

whether a third point is collinear. Mukhopadhyay et al. improved this by finding an algorithm that finds an ordinary line among a set of points in time $O(n \log n)$ [32]. A similar, but simplified, algorithm was demonstrated several years later by Mukhopadhyay and Green [33].

In [34], Lenchner considers the "sharp dual" of this problem, i.e., ordinary intersections determined by an arrangement of lines in $\mathbb{R}^2$, not all parallel and not all passing through a common point. (Since the "sharp dual" is more general than the "dual", the Csima and Sawyer result does not apply.) Lenchner first proved that ordinary intersections occur in such an arrangement, and in fact, that there must exist at least $\frac{5n}{39}$ such points among n lines. He later improved this original result to $\frac{2n-3}{7}$ among $n \geqslant 7$ lines[35].

In the conclusion of [34], Lenchner asks whether an algorithm exists that can find an ordinary intersection in such an arrangement in time $o(n^2)$. The following algorithm performs in time $O(n \log n)$.

3.2.2 Locating an Ordinary Intersection

Definition 3.2.1. Let L_0, L_1, and L_2 be any three lines of $\mathcal{A}$ that intersect at three distinct points. Label the lines such that L_0 and L_1 intersect at point P, which is to the left of the intersection Q of lines L_0 and L_1. Points P and Q are *consecutive points* (on L_0) if no line intersects L_0 on the open interval (P, Q). Furthermore, if P and Q are consecutive points, L_1 and L_2 are *consecutive lines* (with respect to L_0) if L_1 is the "rightmost" line through P and L_2 is the "leftmost" line through Q. In other words, there is no line through P intersecting L_2 at a point closer to Q than $L_1 \cap L_2$, and there is no line through Q closer to P than $L_1 \cap L_2$.

Lemma 3.2.2. *Suppose line L_0 contains no ordinary points. Let X be the closest intersection point above line L_0 incident to at least two lines not parallel to L_0. Then, X must be the intersection of two consecutive lines through two consecutive points P_i and P_{i+1}.*

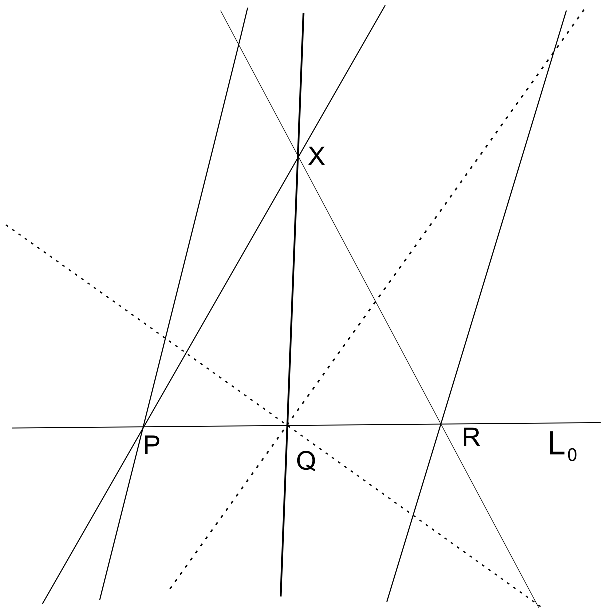

Figure 3.1: A line passing through (non-ordinary) point Q will determine an intersection closer to L_0 than X.

Proof. Suppose X is the intersection of lines, L_1 and L_2, through consecutive points, P_i and P_{i+1}, but the lines are not consecutive. Thus, there exists a line through either P or Q that intersects either L_1 and L_2 at a point closer than X, i.e., a contradiction

So, suppose there are three intersection points, P, Q, and R on L_0 in that order from left to right, and X is the intersection of a line through P and a line through R. Then there is another line through Q that intersects one side of the triangle $\triangle XPR$, interior to the side PX or RX. Either way, there is an intersection point S that is lower than X, i.e, a contradiction. □

Lemma 3.2.3. *Suppose line L_0 contains no ordinary points, and X is the closest intersection point above L_0. Then X is either an ordinary point, or it has a line M through it parallel to L_0.*

Proof. Let P and Q be points in order from left to right on L_0 that contain the lines forming X. (By Lemma 3.2.2, P and Q are consecutive points and X is formed by consecutive lines through these.) Suppose that there is a third line

$\overline{XR}$ through X, where R is another point on L_0. Without loss of generality let's suppose that R is to the right of Q. There is another line through Q that intersects either segment PX or segment RX, and either way X is not the lowest point, i.e., a contradiction. See Figure 3.1. □

Together, Lemmas 3.2.2 and 3.2.3 can be used to prove the dual form of the Sylvester-Gallai Theorem. (A configuration of points can always be dualized such that no two lines are parallel.) From the algorithm below, one can also see proof of its "sharp dual" form.

3.2.3 Algorithm to Find an Ordinary Point in Time $O(n \log n)$

Theorem 3.2.4. *Given an arrangement of n lines in $\mathbb{R}^2$, not all parallel and not all passing through the same point, there exists an algorithm to find a ordinary intersection point in time $O(n \log n)$.*

Proof. Let n be the number of lines in arrangement $\mathcal{A}$. Let L_0, L_1, and L_2 be any three lines of $\mathcal{A}$ that intersect at three distinct points. If no such lines exist then the arrangement consists of only two families of parallel lines and every intersection is ordinary, so suppose this not to be the case. *Time to find L_0, L_1, and L_2: $O(n)$.*

Consider L_0 to be horizontal, and L_1 and L_2 to be intersecting "above" L_0. Find all of the intersection points on L_0. Label them from left to right P_1, P_2, $\ldots$, P_m. *Time to sort them, collecting potentially multiple lines into each P_k: $O(n \log n)$.*

If any P_k is ordinary the algorithm is done, so suppose that none are.

Find the "leftmost" and "rightmost" line through each P_k, i.e., find the pairs of consecutive lines. *Time: $O(deg(P_1) + deg(P_2) + \ldots + deg(P_m)) = O(n)$.*

Let X be the lowest intersection point above L_0, which by Lemma 3.2.2 must be the intersection of consecutive lines through consecutive bundles P_k and P_{k+1}. *Time: $O(n)$.*

Determine whether there is a line M parallel to L_0 that passes through X,

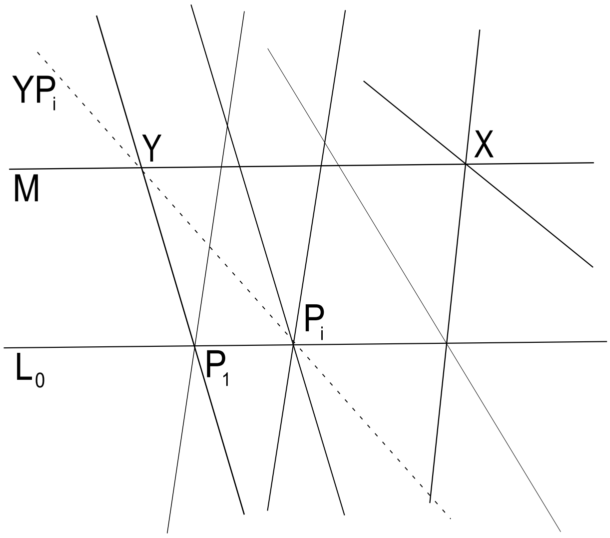

Figure 3.2: If X is the lowest intersection determined by lines not parallel to L_0, then Y must be ordinary.

and if so, then find M. *Time:* $O(n)$. If there is no such line M then, by Lemma 3.2.3, X is an ordinary point.

Otherwise, suppose that M exists. Let Y be the intersection of M with the leftmost line from the leftmost bundle P_1.

Assume Y is not an ordinary point. Then, there exists a point P_k, $k > 1$, such that YP_k is a line of the arrangement, and there is another line through P_1 that intersects YP_k in its interior at a point lower than Y and therefore lower than X, i.e., a contradiction. Hence Y is an ordinary point. (See Figure 3.2.) *Time to find* Y: $O(1)$. □

3.3 Ordinary Points in an Arrangement of Hyperplanes in $\mathbb{R}^d$

3.3.1 Duality

Given n points in $\mathbb{R}^3$, no three on a line and not all on a plane, does there necessarily exist a three-point plane? Recently, the present authors proved that, under the same hypothesis, there must exist at least $\frac{4}{13}\binom{n}{2}$ such planes [36]. Without the assumption that no three points are collinear, Bonnice and Kelly showed that there must exist at least $\frac{3n}{11}$ ordinary planes in 3-space [37]. The existence of such a plane also follows from Hansen's result on the existence of *ordinary* hyperplanes in d-dimensional projective space [38]. (In 3-space their existence was proved earlier by Motzkin, but for 4 or higher it's due to Hansen.) An ordinary hyperplane in d-space is one in which all but one of the points lie on a $(d-2)$-flat.

The algorithm presented in Section 3.3.2 solves the following problem. Find the intersection point of exactly d hyperplanes in $\mathbb{R}^d$ (i.e., an ordinary intersection) in an arrangment of hyperplanes, not all parallel, not all passing through the same point, and no d passing through a line. This problem on hyperplanes is the "sharp dual" of a problem on points. That is, given a set of n points in $\mathbb{R}^d$, no d on a $(d-2)$-flat and not all on a hyperplane, find a hyperplane determined by exactly d points. (It follows from this hypothesis that no k points lie on a $(k-2)$-flat for $3 \leqslant k \leqslant d$.)

For the convenience of the reader, we provide the following correspondences between flats and their duals:

- Hyperplanes $\longleftrightarrow$ Points
- $(d-2)$-flats $\longleftrightarrow$ Lines
- k-flats $\longleftrightarrow$ $(d-k-1)$-flats

3.3.2 Algorithm to Find an Ordinary Point in Time $O(n \log n)$

Before we claim to have an algorithm to find an ordinary intersection, we must first be sure that a given set of hyperplanes determines an intersection point. Let $h_i^{\perp}$ be a normal vector to the hyperplane h_i. The following lemma shows the necessary and sufficient conditions for a general intersection point to exist.

Lemma 3.3.1. *Let $H = \{h_0, h_1, \ldots, h_{d-1}\}$ be a set of d hyperplanes in $\mathbb{R}^d$. The* *hyperplanes of H determine an intersection point if and only if their normals form a basis for $\mathbb{R}^d$, i.e., span$(h_0^\perp, h_1^\perp, \ldots, h_{d-1}^\perp\}) = \mathbb{R}^d$.*

Proof. This follows from the observation that the space orthogonal to the intersection of hyperplanes is the span of the hyperplanes' normals. That is, given a set of k hyperplanes $H' = \{h'_0, h'_1, \ldots, h'_{k-1}\}$, then $(h'_0 \cap h'_1 \cap \ldots \cap h'_{k-1})^\perp = span(h_0'^\perp, h_1'^\perp, \ldots, h_{k-1}'^\perp)$. (Note that the sum of the dimension of a space and the dimension of its orthogonal space in $\mathbb{R}^d$ is always d). □

Given a set of k vectors one can find a maximal linearly independent subset in time $O(k^2)$ using a method such as row reduction on a matrix. By letting $k = n$, we could determine such a subset (i.e., a basis for the span) of n vectors, for any n, in time $O(n^2)$. However, one can do better.

Lemma 3.3.2. *A maximal linearly independent subset from a set of n vectors can be found in time $O(n)$.*

Proof. We assume the dimension d to be constant, and thus, a maximal linearly independent set of d vectors can be determined in constant time.

Let $S = \{v_0, v_1, \ldots, v_{n-1}\}$ be the set of n vectors from which a maximal linearly independent set must be found. Let M be a matrix of dimension $d \times d$ with rows initialized to the vectors $v_0, v_1, \ldots, v_{d-1}$. Use row reduction, tracking the vector corresponding to each row (e.g., updating as needed upon a row exchange), to determine a linearly independent subset of these vectors. *Time* $O(1)$.

If these vectors span $\mathbb{R}^d$ then the algorithm may terminate. Otherwise, discard the zero rows from the row reduced M, replacing them with vectors from S while still maintaining a correspondence between rows and vectors. *Time* $O(1)$.

Begin the next iteration by performing a row reduction on M to find a linearly independent set. (Note that one needs only to reduce the rows of M that were most recently added.) After the row reduction on each iteration, zero rows are replaced with vectors from S. The iterating continues until either S is exhausted or a linearly independent set of size d is found. This repeats at most $n - d$ times. *Time* $O(n)$.

The resulting set of vectors forms a maximal linearly independent set. □

The following lemma demonstrates the strength of the hypothesis needed by our algorithm.

Lemma 3.3.3. *Given that no d hyperplanes of a set in $\mathbb{R}^d$ pass through the same line, no k hyperplanes of that set contain the same $(d - k + 1)$-flat for $3 \leqslant k \leqslant d$.*

Proof. Suppose hyperplanes $h_0, h_1, \ldots, h_{k-1}$ all contain a $(d-k+1)$-flat, then $dim(h_0 \cap h_1 \cap \ldots \cap h_{k-1}) \geqslant d - k + 1$. Thus, by intersecting with an additional $(d-k)$ hyperplanes, for a total of d hyperplanes, (assuming no two are parallel) the resulting flat will have dimension at least $(d - k + 1) - (d - k) = 1$, and $dim(h_0 \cap h_1 \cap \ldots \cap h_{d-1}) \geqslant 1$, i.e., a contradiction. (If any two hyperplanes in the intersection are parallel, the result is an empty set.) □

We will assume that the intersection of two flats can be found in constant time.

Theorem 3.3.4. *Given an arrangement of n hyperplanes in $\mathbb{R}^d$, not all parallel, not all passing through the same point, and no d passing through a line, there exists an algorithm to find an ordinary intersection point, or determine that none exists, in time $O(n \log n)$.*

Proof. Let H be the set of n hyperplanes, $\{h_0, h_1, \ldots, h_{n-1}\}$, in $\mathbb{R}^d$, $n \geqslant d$, not all parallel, no d passing through a line and not all passing through the same point.

For each hyperplane, compute its normalized normal vector. The first non-zero coordinate of each normal vector should be positive (replacing the vector by its negative if necessary), so that two hyperplanes are parallel if and only if their normal vectors are the same. *Time:* $O(n)$.

Sort the hyperplanes, lexicographically, by their normals into k families of parallel hyperplanes. That is, let $H^{(0)}$, $H^{(1)}$, $\ldots$, $H^{(k-1)}$ each be a set of hyperplanes such that for any two $h \in H^{(i)}$, $h' \in H^{(j)}$, h and h' are parallel if and only if $i = j$. Since parallelism is an equivalence relation, this forms a partition on the set H. So, obviously, $|H^{(0)}| + |H^{(1)}| + \ldots + |H^{(k-1)}| = |H| = n$. Let $h_i^{(j)}$, for $0 \leqslant i < |H^{(j)}|$, be the members of the set $H^{(j)}$. *Time:* $O(n \log n)$.

For each set of hyperplanes, $H^{(i)}$, there is a distinct normal. Use the algorithm described in Lemma 3.3.2 to find a maximal linearly independent set from these normal vectors, tracking for each normal the associated hyperplane family. If the maximal linearly independent set does not span $\mathbb{R}^d$, then by Lemma 3.3.1 there will exist no intersection point, and the algorithm is finished *Time:* $O(n)$.

From now on we assume that an intersection exists, and therefore the number of hyperplane families, k, is at least d.

Let M be the plane formed by intersecting a member from each of the first $d-2$ sets of hyperplanes from the d sets whose normals formed the basis in the previous step. Without loss of generality, we will assume that these $d-2$ families are $H^0, H^1, \ldots, H^{d-3}$. Thus, $M = h_0^{(0)} \cap h_0^{(1)} \cap \ldots \cap h_0^{(d-3)}$. By Lemma 3.3.3, $dim(M) = 2$. *Time to determine* M*:* $O(n)$.

Let L be the lines formed by intersecting each of the remaining hyperplane families with M, i.e., $L = \{l_i = M \cap h_i : h_i \in \left(H^{(d-2)} \cup H^{(d-1)} \cup \ldots \cup H^{(k-1)}\right)\}$, where each l_i is a line. (Note that the hyperplanes used to form lines on M all intersect M since they are not parallel to any of the hyperplanes used in the construction of M.) *Time to determine the set* L*:* $O(n)$.

Consider the following cases.

Case 1: The lines of L are all parallel on M.

Consider these hyperplanes in projective space. There exists a point on the hyperplane at infinity, p_∞, that is incident to all lines of L, and thus, incident to all hyperplanes in $\{H^{(d-2)}, H^{(d-1)}, \ldots, H^{(k-1)}\}$. Furthermore, p_∞ is incident to all hyperplanes in $\{H^{(0)}, H^{(1)}, \ldots, H^{(d-3)}\}$, since M and all of its constituent hyperplanes, and thus, the parallel hyperplane families, pass through p_∞. Since the hyperplanes of H all share a common point at infinity, there will be no finite intersection point. However, since we showed that the hyperplane normals spanned $\mathbb{R}^d$, this case is not possible.

Case 2: The lines of L are all concurrent on M, and there are at least three of them.

For this case, we know that $|H^{(d-2)}| = |H^{(d-1)}| = \ldots = |H^{(k-1)}| = 1$. Let p be the point on M at which the lines of L all cross. *Time to determine if all concurrent: $O(n)$.*

Since the lines of L all pass through p, we will construct another plane M' (parallel to M) by using one alternative member from one of the first $d-2$ parallel families.

If no alternative member exists, then all hyperplane families contain just one member. Thus, all hyperplanes pass through point p, in violation of the hypothesis. In this case, no ordinary point exists and the algorithm terminates.

Without loss of generality, assume $h_1^{(0)}$ (i.e., a second member from the set $H^{(0)}$) is the alternative member used to construct M'. That is, let $M' = h_1^{(0)} \cap h_0^{(1)} \cap \ldots \cap h_0^{(d-3)}$. *Time to construct M': $O(n)$.*

Construct the set L' in an analogous manner to the construction of L, i.e., let $L' = \{l'_i = M' \cap h_i : h_i \in \left(H^{(d-2)} \cup H^{(d-1)} \cup \ldots \cup H^{(k-1)}\right)\}$. The lines of L' cannot all be parallel by the same argument used in Case 1.

Assume the lines of L' are again all concurrent at a finite point p'. Then the line determined by p and p', i.e. $\overline{pp'}$, is contained in the $d-3$ hyperplanes used to construct both M and M', which excludes the two hyperplanes used from the set $H^{(0)}$. (An intersection of hyperplanes containing two distinct points, also contains the line that connects them.) That is, $\overline{pp'}$ is contained in $h_0^{(1)}, h_0^{(2)}, \ldots, h_0^{(d-3)}$. The line $\overline{pp'}$ is also contained in the three or more hyperplanes that formed the lines of L and L'. Altogether, there must be at least $(d-3)+3 = d$ hyperplanes containing the line $\overline{pp'}$, in violation of our hypothesis. *Time to determine if all concurrent on M': $O(n)$.*

Therefore, M' contains an ordinary intersection, and we may proceed to the next case, mutatis mutandis.

Case 3: The lines of L form an ordinary intersection on M.

The ordinary intersection formed by the lines of L can be found using the algorithm given in Section 3.2.3. Assume l_i and l_j form an ordinary intersection on M. This point is the intersection of the hyperplanes $h_0^{(0)} \cap h_0^{(1)} \cap \ldots \cap h_0^{(d-3)}$ and exactly two other hyperplanes (which formed l_i and l_j), and thus, at the intersection of d hyperplanes. Therefore, we have found an ordinary intersection. *Time:* $O(n \log n)$. □

3.4 Arrangements of Pseudolines

3.4.1 Ordinary Intersection Points

Now consider an arrangement of pseudolines, any two of which cross and not all at the same point. Any such arrangement contains an ordinary intersection, and the best result in this area is that of Csima and Sawyer [27], who extended their $\frac{6n}{13}$ result to also include ordinary intersections among pseudolines in the projective plane. An elegant proof of the existence of ordinary intersections can be found using Euler's formula to find an inequality due to Melchior [18]. See Felsner's book [15] for excellent coverage of this and other results related to Sylvester's Problem.

Arrangements of pseudolines, as discussed in this paper, have certain properties that are assumed:

- Each pseudoline goes off to infinity in both directions.
- No pseudoline crosses itself.
- Each pair of pseudolines intersects at exactly one point, and at that point cross.
- More than two pseudolines may cross at a single point (otherwise the intersection is ordinary).

- The pseudolines do not all cross at the same point (i.e., there is more than one intersection point).

See [15] for a more complete explanation of pseudoline arrangements and their properties.

It is assumed that given a point P and a pseudoline L, one can determine whether P lies on L in time $O(1)$. Therefore, in an arrangement of n pseudolines, the pseudolines that cross P can be determined in time $O(n)$. It is also assumed that the intersection point of any two pseudolines can be found in time $O(1)$.

Recently a couple of results related to the following algorithm have been published or submitted. Pretorius and Swanepoel in [39] provide proof of a theorem that generalizes both Sylvester-Gallai and Motzkin-Rabin. Their proof utilizes a sequence of successively smaller triangles that terminates with finding the desired intersection point. A similar method is also used by Lenchner in [40]. Note that one might also see similarity between these proofs and Motzkin's as published in [23] (i.e. they utilize what Motzkin calls "characteristic triangles").

The algorithm described below was inspired by the recent proof given by Lenchner in [40]. This algorithm can be used to find ordinary point in an arrangement of lines, and by duality an ordinary line determined by a set of points. We must also mention that a $O(n^2)$ algorithm could be obtained by an incremental construction of the arrangement that tracks the intersections that are created. However, such an algorithm would not also prove the existence of an ordinary intersection point.

3.4.2 A $O(n^2)$ Algorithm to Find an Ordinary Point[2]

Theorem 3.4.1. *An ordinary intersection can be found in an arrangement of pseduolines in time $O(n^2)$.*

Proof. Let L_0, L_1 and L_2 be any three pseudolines of arrangement $\mathcal{A}$ that intersect at three distinct points. *Time to find three such lines: O(n).*

[2]The results of this section should primarily be attributed to my advisor, George Purdy.

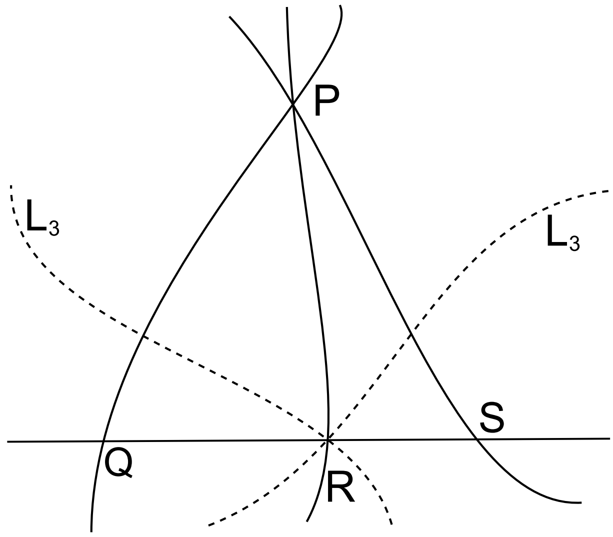

Figure 3.3: If point R is not ordinary, then a third pseudoline L_3 must cross either segment PQ or PS.

Let P be the intersection point of L_1 and L_2. If P is ordinary, then the algorithm is done. *Time to determine whether P is ordinary: O(n).*

Otherwise, there are at least three pseudolines (L'_1, L'_2 and L'_3) crossing at P. Consider L_0 to be "horizontal" and P "above" L_0. Let points Q, R and S be the points of intersection left to right on L_0 of the three pseudolines crossing at P (i.e., L'_1, L'_2 and L'_3). If R is an ordinary intersection, then the algorithm is done. *Time to determine whether R is ordinary: O(n).*

Otherwise there is a pseudoline, L_3, crossing at R that either crosses the finite segment QP or PS. *Time to determine where* L_3 *crosses:* $O(1)$.

Without loss of generality, assume this pseudoline crossing R also crosses QP. This pseudoline is defined to be the triangle's *dividing line*. The configuration will now be reoriented for recursion, letting R be the intersection of L_3 with pseudoline QP, P the previous R, Q the previous P, and S the previous Q. *Time to reorient the configuration for recursion:* $O(1)$.

The following lemma states that this recursion repeats no more than n times, yielding a time $O(n^2)$ algorithm. □

Lemma 3.4.2. *No pseudoline is used by the algorithm as a* dividing line *more* *than once.*

Proof. Each *dividing line* L crosses the interior of a triangle, dividing it into two parts. All subsequent *dividing lines* used by the algorithm must cross the interior of one of those parts, of which L lies on the boundary. □

This second algorithm has a potential advantage over our first since it may stop early, possibly at the first intersection P (i.e. time $\Omega(n)$). By duality, this algorithm also can be used to find ordinary lines in an arrangement of points, with the same time complexity.

3.4.3 Existence of Monochromatic Points in a Bichromatic Arrangement

In a bichromatic arrangement of pseudolines, any two crossing and not all crossing at the same point, a monochromatic intersection point always exists, but it might not exist for both colors. An arrangement containing monochromatic intersections of only one color is called "biased" (see [31]). The existence of biased arrangements requires any algorithm in search of a monochromatic intersection to consider both colors (or at least be run twice if limited to a specific color).

The previous algorithm will now be modified to find a monochromatic intersection. While Chakerian [29] and others have proven that lines in the real projective plane always determine a monochromatic intersection (and an argument similar to theirs might be extended to include pseudolines), the present authors are unaware of a proof that explicitly extends this result to pseudolines in the euclidean plane. The algorithm below provides such a proof.

In [41], Pretorius and Swanepole provide an algorithm (i.e. an algorithmic proof) to find a monochromatic line in a bichromatic set of points, apparently in time $O(n^2)$ (although the present authors are unaware of a "worst-case" instance for their algorithm). Note that the bichromatic pseudoline problem is more general than that of points, since not every pseudoline arrangement has a dual.

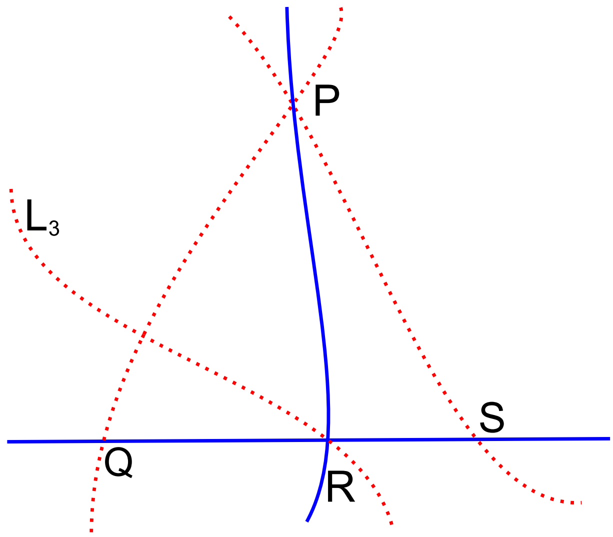

Figure 3.4: If point R is monochromatic, then a third pseudoline L_3 with a different color must cross either segment PQ or PS.

As with the previous algorithm, it is assumed that given a point P and a pseudoline L, one can determine whether P lies on L in time $O(1)$. Therefore, in an arrangement of n pseudolines, the pseudolines that cross P can be determined in time $O(n)$. It is also assumed that the intersection point of any two pseudolines can be found in time $O(1)$.

3.4.4 Algorithm to Find a Monochromatic Intersection in a Bichromatic Arrangement of Pseudolines[3]

Theorem 3.4.3. *A monochromatic intersection in a bichromatic arrangement of pseudolines may be found in time $O(n^2)$.*

Proof. Let L_0 be a pseudoline from an arrangement, $\mathcal{A}$, containing n pseudolines each colored one of red or blue, any two of which cross but not all cross at the same point. Consider L_0 to be "horizontal" and, without loss of generality, assume its color is blue. *Time:* $O(1)$.

Let Q and S be the leftmost and rightmost intersection points on L_0. Assume a red pseudoline crosses at Q, another red pseudoline crosses at S, and let

[3]The results of this section should primarily be attributed to my advisor, George Purdy.

P be their intersection point "above" L_0. If this assumption is false (i.e. red pseudolines do not cross both Q and S), then at least one of Q or S is monochromatic and the algorithm is done. *Time to find Q and S and determine whether they are monochromatic: $O(n)$.*

If P, the intersection of the red pseudolines crossing Q and S, is monochromatic then again, the algorithm is done. Otherwise, a blue pseudoline L_2 crosses P and intersects L_0 at point R, between Q and S. See Figure 3.4.

At this point, the "setup" is complete and we begin the first step of a (potentially) recursive process to find a monochromatic intersection.

If R it is monochromatic (blue), then the algorithm is done. *Time to determine whether R is monochromatic: $O(n)$.*

Otherwise, a red pseudoline, L_3, crosses R and intersects either segment PQ or segment PS. Without loss of generality, assume it crosses PQ. This pseudoline is defined to be the triangle's (i.e. $\triangle PQR$'s) *dividing line.* The configuration will now be reoriented for recursion, letting R be the intersection of L_3 with pseudoline QP, P the previous R, Q the previous P, and S the previous Q. *Time to reorient the configuration for recursion: $O(1)$.*

Note that each step of the recursive process, expects R to possess a different, possibly monochromatic, color. So for the first and all other odd numbered steps it would expect "blue", and likewise "red" for the even. Again, we refer to Lemma 3.4.2 to show that this algorithm runs in time $O(n^2)$. □

Chapter 4

Bichromatic and Equichromatic Lines

4.1 Introduction

Questions concerning "how many?" or "what types?" of lines are determined by a set of points have been asked since (no later than) 1893 when J.J. Sylvester, in [10], essentially asked whether any non-collinear set of points necessarily determines an ordinary line (i.e. a line that passes through exactly two points). Unaware that it had previously been raised, Erdös arrived at the same question (about 40 years later) while attempting to prove by induction that there are at least n lines determined by a set of n points. T. Gallai provided a proof in the affirmative which was published by Steinberg in [11], hence it is now called the Sylvester-Gallai Theorem. Interestingly, an inequality due to Melchior also provides proof of this theorem, and was published four years earlier in [18].

More directly related to this article is a question first asked by R.L. Graham around 1965 (see [31]) of whether a bichromatic arrangement of lines necessarily determines a monochromatic point (the dual of the context for the present article). A few years passed before the first proof was published, again in the affirmative, by G.D. Chakerian in [29]. However, Chakerian is not credited with the first proof. That honor belongs to T.S. Motzkin and M. Rabin (and in his article, Chakerian acknowledges this). The theorem is now commonly called the Motzkin-Rabin Theorem.

Motivated by a conjecture of Fukuda [28] (now known to be false for a specific case with nine points [2]), in their groundbreaking paper Pach and Pinchasi demonstrated several lower bounds on the number of bichromatic lines incident to few points [30] (see sections 4.2.3 and 4.3.2 for corrections and improvements to a result from this paper). A line is bichromatic when it passes through at least one point of each color, and monochromatic otherwise. (Note that only lines incident to at least two points are considered.) The best lower bounds established in [30] concern the case where there is an equal number of points of each color, i.e $|G| = |R|$. The present article provides improved lower bounds for a subset of the bichromatic lines, called equichromatic. Several lower bounds are shown to also be true for lines determined by points in the complex plane ($\mathbb{C}^2$).

In [42], Kleitman and Pinchasi consider the case where neither color class, G nor R, is collinear. They conjecture that among all such arrangements there are at least $|G \cup R| - 1$ bichromatic lines, assuming $|G| = n$ and $n-1 \leqslant |R| \leqslant n$, and they prove a lower bound of $|G \cup R| - 3$ bichromatic lines. The present article will prove their conjecture for sufficiently large n.

Pach and Pinchasi derive their results from two identities, which are used again in [42]. Let $t_{i,j}$ be the number of lines which pass through exactly i green and j red points. With the assumption that $|G| = |R| = n$, they show that the number of bichromatic point pairs (i.e. n^2) is equal to a summation over all lines determined by the sets:

$$\sum_{\substack{i,j \geqslant 0 \\ i+j \geqslant 2}} ij t_{i,j} = n^2.$$

Simlarly, monochromatic pairs of points can be counted as:

$$\sum_{\substack{i,j \geqslant 0 \\ i+j \geqslant 2}} \left[\binom{i}{2} + \binom{j}{2}\right] t_{i,j} = 2\binom{n}{2} = n^2 - n.$$

More generally, if one assumes $|G| = n$ and $|R| = n - k$, these identities become:

$$\sum_{\substack{i,j \geqslant 0 \\ i+j \geqslant 2}} ij t_{i,j} = n(n-k) = n^2 - nk \tag{4.1}$$

and,

$$\begin{aligned}\sum_{\substack{i,j \geqslant 0 \\ i+j \geqslant 2}} \left[\binom{i}{2} + \binom{j}{2}\right] t_{i,j} \\ &= \frac{n(n-1)}{2} + \frac{(n-k)(n-k-1)}{2} \\ &= n^2 - n - nk + \frac{k^2 + k}{2}. \end{aligned} \tag{4.2}$$

By subtracting (4.1) from (4.2) and then splitting the summation, the following identity is found:

$$\sum_{\substack{i,j \geqslant 0 \\ i+j \geqslant 2}} (i+j) t_{i,j} = \sum_{\substack{i,j \geqslant 0 \\ i+j \geqslant 2}} (i-j)^2 t_{i,j} + 2n - (k^2 + k). \tag{4.3}$$

This identity will be used throughout the present article.

4.2 Equichromatic Lines

4.2.1 Lower Bound in $\mathbb{R}^2$

Definition 4.2.1. Any line passing through i green and j red points such that $i + j \geqslant 2$ and $|i - j| \leqslant 1$ is called *equichromatic.*

An *equichromatic line* can be thought of as one in which the number of points of each color, lying on the line, are as "equal" as possible, i.e. equal if the line contains an even number of points, and otherwise differing by only one. Let Q be the set of equichromatic lines determined by $G \cup R$, and let B be the set of bichromatic lines. A lower bound on the number of equichromatic lines is demonstrated below, and since $Q \subseteq B$, this also applies to bichromatic lines.

Let t_k be the number of lines which pass through exactly k points. In [18], Melchior published the following inequality (which follows from Euler's polyhedral formula):

$$\sum_{k \geqslant 2} (k-3)t_k \leqslant -3.$$

Within our context, this can be rewritten as:

$$\sum_{\substack{i,j \geqslant 0 \\ i+j \geqslant 2}} (i+j)t_{i,j} \leqslant 3 \cdot \sum_{\substack{i,j \geqslant 0 \\ i+j \geqslant 2}} t_{i,j} - 3. \tag{4.4}$$

By applying to this the identity above, (4.3), and then reuniting the two summations this becomes:

$$\sum_{\substack{i,j \geqslant 0 \\ i+j \geqslant 2}} ((i-j)^2 - 3)t_{i,j} \leqslant -2n - 3 + (k^2 + k).$$

Let q_m be the the sum of all $t_{i,j}$ such that $|i-j| = m$, that is $q_m = \sum_{|i-j|=m} t_{i,j}$. Note that the number of equichromatic lines (i.e. $|Q|$) is equal to q_0+q_1. Let t be the total number of lines determined by $G \cup R$. Now, by partially unwinding the summation and then negating the inequality, it becomes:

$$3q_0 + 2q_1 \geqslant 2n + 3 + q_2 + 6q_3 + \sum_{m \geqslant 4} (m^2 - 3)q_m - (k^2 + k)$$

$$3 \cdot |Q| \geqslant 2n + 3 + (t - |Q|) - (k^2 + k).$$

This gives us:

Theorem 4.2.2. *Let G and R each be a finite set of green or red points, respectively, in $\mathbb{R}^2$ such that $|G| = n$, $|R| = n - k$, $G \cap R = \emptyset$, and the points* *of $G \cup R$ are not all collinear. Let t be the total number of lines determined by $G \cup R$. The number of equichromatic lines is at least $\frac{1}{4}(t + 2n + 3 - k(k+1))$.*

Using the Erdös-de Bruijn Theorem, i.e. $t \geqslant 2n - k$, one can see the following corollary:

Corollary 4.2.3. *The number of equichromatic lines, $|Q|$, is at least $n + \frac{1}{4}(3 - k(k+2))$. If $k \in \{0, 1\}$, then $|Q| \geqslant n + 1 - k$.*

4.2.2 Proof of the Kleitman-Pinchasi Conjecture

In [42], Kleitman and Pinchasi conjectured that when neither color class is collinear there exist at least $2n - k - 1$ bichromatic lines, assuming $|G| = n$, $|R| = n - k$ and $k \in \{0, 1\}$. Since equichromatic lines are a subset of bichromatic, our lower bound (i.e. Theorem 4.2.2) is better than the Kleitman-Pinchasi conjecture whenever $t \geqslant 6n - 7$ (or $t \geqslant 6n - 9$ if $k = 1$). We now prove that when n is sufficiently large, their conjecture is true.

When all but a few points lie on a line, one can verify the conjectured bound directly. We give the following lemmas:

Lemma 4.2.4. *If neither color class is collinear, and $2n - k - 2$ points are incident to one line then $|Q| \geqslant 2n - k - 1$.*

Lemma 4.2.5. *If neither color class is collinear, and $2n - k - 3$ points are incident to one line then $|Q| \geqslant 3n - k - 4$.*

(Note that by Lemma 4.2.4 the conjectured $2n - k - 1$ lower bound is sharp, i.e. it cannot be improved.)

Although one can count the number of equichromatic lines in the specific cases above, a tool is needed for more general point configurations. Just such a tool can be found in a well known paper from 1958 by Kelly and Moser.

In [25], Kelly and Moser proved a lower bound on the number of lines, t, assuming at most $2n - k - r$ points are collinear. (Note that Kelly and Moser originally assumed n points total, but the present article uses the context of $2n - k$ points.) Their lower bound is as follows:

Theorem 4.2.6. *If at most $2n-k-r$ points are incident to a single line and $2n-k \geqslant \frac{1}{2}(3(3r-2)^2+3r-1)$ then:*

$$t \geqslant r(2n-k) - \frac{(3r+2)(r-1)}{2}.$$

So by letting $r=4$ in the inequality above, we get the needed lower bound for the total number of lines:

Lemma 4.2.7. *If $n \geqslant 78+k$ and no more than $2n-k-4$ points are incident to one line then $t \geqslant 8n-4k-21$.*

By Lemmas 4.2.4, 4.2.5, and 4.2.7 we have the following theorem:

Theorem 4.2.8. *Let G and R each be a finite set of green or red points, respectively, in $\mathbb{R}^2$ such that $|G|=n$, $|R|=n-k$, $k \in \{0,1\}$, $G \cap R = \emptyset$, and neither color class is collinear. If $n \geqslant 78+k$, then the number of equichromatic lines is at least $2n-k-1$.*

Thus, the Kleitman-Pinchasi conjecture is true for all $n \geqslant 79$.

4.2.3 Equichromatic Lines With Few Points

Equichromatic Lines Through At Most Four Points

Although Pach and Pinchasi did not define "equichromatic", Theorem 2(i) in their paper [30] proves a lower bound on the number of equichromatic lines passing through at most four points. For the convenience of the reader, we will show Pach and Pinchasi's derivation except we will assume $|G|=n$ and $|R|=n-k$ (i.e. we do not assume $|G|=|R|$ as was originally).

By adding twice Melchior's inequality (4.4) to the identity (4.3), one can see that:

$$\sum_{\substack{i,j \geqslant 0 \\ i+j \geqslant 2}} \left(6-(i+j)-(i-j)^2\right) t_{i,j} \geqslant 2n+6-k(k+1).$$

Let $\gamma_{i,j}$ be the coefficient corresponding to $t_{i,j}$ in the summation above. The only positive coefficients are $\gamma_{1,1}=4$ and $\gamma_{1,2}=\gamma_{2,1}=\gamma_{2,2}=2$. So, it must be the case that:

$$2t_{1,1} + t_{1,2} + t_{2,1} + t_{2,2} \geqslant n + 3 - \frac{k(k+1)}{2}.$$

From this we get the following theorem:

Theorem 4.2.9. *Let G and R each be a finite set of green or red points, respectively, in $\mathbb{R}^2$ such that $|G| = n$, $|R| = n - k$, $G \cap R = \emptyset$, and the points of $G \cup R$ are not all collinear. The number of equichromatic lines determined by at most four points is at least $\frac{1}{4}(2n + 6 - k(k+1))$.*

Equichromatic Lines Through At Most Five Points

To extend these results to equichromatic lines in the complex plane, Hirzebruch's "Improved" Inequality will be used. The Hirzebruch Inequalities are derived from the Miyaoka-Yau Inequality (known in the field of algebraic geometry) and they are true in the complex plane (i.e. $\mathbb{C}^2$). Unlike Melchior's Inequality, both of Hirzebruch's Inequalities show that among a set of points in $\mathbb{C}^2$ there must exist a line determined by at most *three* points (see also [23] and [43]). One must note that these inequalities were originally proven for the dual of the present context, i.e. an arrangement of lines in the complex plane (such that t_k was the number of intersection points at which k lines cross). The present article will instead remain consistent with the context used by Kleitman, Pach and Pinchasi. Of course, the lower bound established below for points in $\mathbb{C}^2$ also applies to points in $\mathbb{R}^2$.

The first of Hirzebruch's two inequalities, as used in the present article, was published in [44]. It states that among a set of n points in $\mathbb{C}^2$ and assuming $t_n = t_{n-1} = 0$:

$$t_2 + t_3 \geqslant n + \sum_{k \geqslant 5} (k-4)t_k. \tag{4.5}$$

In the slightly more restrictive case of $t_n = t_{n-1} = t_{n-2} = 0$, published in

[45], there exists an improved inequality:

$$t_2 + \frac{3}{4}t_3 \geqslant n + \sum_{k\geqslant 5}(2k-9)t_k. \tag{4.6}$$

In the present context (i.e. $|G| = n$, $|B| = n - k$), the improved inequality can be rewritten:

$$\begin{aligned} &-(t_{0,2}+t_{2,0}) - t_{1,1} - \frac{3}{4}(t_{0,3}+t_{3,0}) - \frac{3}{4}(t_{1,2}+t_{2,1}) \\ &\qquad + \sum_{\substack{i,j\geqslant 0\\ i+j\geqslant 5}} (2(i+j)-9)t_{i,j} \leqslant -(2n-k). \end{aligned} \tag{4.7}$$

Similarly, subtracting (4.1) from (4.2) and unwinding the first few terms of the summation produces:

$$\begin{aligned} &(t_{0,2}+t_{2,0}) - t_{1,1} + 3(t_{0,3}+t_{3,0}) - (t_{1,2}+t_{2,1}) + 6(t_{0,4}+t_{4,0}) - 2t_{2,2} \\ &\qquad + \sum_{\substack{i,j\geqslant 0\\ i+j\geqslant 5}} \left(\binom{i}{2} + \binom{j}{2} - ij\right) t_{i,j} = -n + \frac{k^2+k}{2}. \end{aligned} \tag{4.8}$$

By adding two times (4.8) and $(1+\epsilon)$ times (4.7), it becomes:

$$\begin{aligned} &(1-\epsilon)(t_{2,0}+t_{0,2}) - (3+\epsilon)t_{1,1} + (\frac{21}{4} - \frac{3}{4}\epsilon)(t_{0,3}+t_{3,0}) \\ &\qquad - (\frac{11}{4} + \frac{3}{4}\epsilon)(t_{1,2}+t_{2,1}) + 12(t_{0,4}+t_{4,0}) - 4t_{2,2} \\ &\qquad + \sum_{\substack{i,j\geqslant 0\\ i+j\geqslant 5}} \big((i-j)^2 + \epsilon(2(i+j)-9) + i + j - 9\big)\, t_{i,j} \\ &\qquad\qquad \leqslant -2n(2+\epsilon) + k(k+2+\epsilon). \end{aligned} \tag{4.9}$$

Let $\epsilon = 1$ and let $\gamma_{i,j}$ be the coefficient for $t_{i,j}$ produced by the left side of the inequality above. One can see that the only negative coefficients are $\gamma_{1,1} = \gamma_{2,3} = \gamma_{3,2} = -2$, $\gamma_{1,2} = \gamma_{2,1} = -\frac{7}{2}$ and $\gamma_{2,2} = -4$, so:

$$-4 \cdot (t_{1,1} + t_{1,2} + t_{2,1} + t_{2,2} + t_{2,3} + t_{3,2}) \leqslant -6n + k(k+3).$$

This gives us:

Theorem 4.2.10. *Let G and R each be a finite set of green or red points, respectively, in $\mathbb{C}^2$ such that $|G| = n$, $|R| = n-k$, $G \cap R = \emptyset$, and no $2n-k-2$ points of $G \cup R$ are collinear. The number of equichromatic lines determined by at most five points is at least $\frac{1}{4}(6n - k(k+3))$.*

For the cases where at most $2n-k-2$ points are collinear, we provide the following two lemmas:

Lemma 4.2.11. *If exactly $2n-k-2$ points of $G \cup R$ are collinear in $\mathbb{C}^2$, the number of equichromatic lines, $|Q|$, determined by at most three points is at least $2(n-k-1)$. Since the total number of lines, t, is at least $2(2n-k-2)$ and at most $2(2n-k-1)$, $|Q| \geqslant (t-2k-2)/2$.*

Lemma 4.2.12. *If exactly $2n-k-1$ points of $G \cup R$ are collinear in $\mathbb{C}^2$, the number of equichromatic lines, $|Q|$, determined by at most two points is at least $n-k$. Since the total number of lines, t, is $2n-k$, $|Q| \geqslant (t-k)/2$.*

Note that the bound given in Lemma 4.2.11 is better than that of Theorem 4.2.10 for all $n \geqslant 7$. Thus, if at most $2n-k-2$ points are collinear and $n \geqslant 7$, then the number of equichromatic lines is at least $\frac{1}{4}(6n - k(k+3))$. (These lemmas will be referred to again to augment results in the complex plane.)

Equichromatic Lines Through At Most Six Points

In [30], Pach and Pinchasi claim "the number of bichromatic lines passing through at most six points is at least one tenth of the total number of connecting lines". Although this statement is true (the present article contains an even better result of $\frac{t}{7} + n$), their derivation contains a small mistake (i.e. Theorem 2(ii) of their paper should state "eight" points instead of "six", since coefficient $\gamma_{4,4} = 0$, not $\frac{2}{5}$ as would have been needed). Below, we show a fix for their derivation which maintains the "at most six points", applies it to equichromatic lines, but weakens the result to a twelfth of the total number of lines. Also, see Section 4.3.2 for a stronger result for bichromatic lines through at most six points in $\mathbb{C}^2$ (which of course applies to $\mathbb{R}^2$).

Again following the method of Pach and Pinchasi, add the identity (4.3) to $(1+\epsilon)$ times (4.4). From this one obtains (the negated form of what was used by Pach and Pinchasi):

$$\sum_{\substack{i,j\geqslant 0\\ i+j\leqslant 2}} \left(3-\epsilon(i+j-3)-(i-j)^2\right) t_{i,j} \geqslant 2n+3(1+\epsilon)-k(k+1).$$

Let $\epsilon = \frac{2}{3}$ and $\gamma_{i,j}$ be the coefficient for $t_{i,j}$ in the summation above. With careful inspection, one can verify that the only positive coefficients are $\gamma_{2,3} = \gamma_{3,2} = \frac{2}{3}$, $\gamma_{3,3} = 1$, $\gamma_{1,2} = \gamma_{2,1} = 2$, $\gamma_{2,2} = \frac{7}{3}$, and the largest $\gamma_{1,1} = \frac{11}{3}$. Of the non-positive coefficients, $\gamma_{0,2} = -\frac{1}{3}$ is the largest. So by subtracting $4\cdot(t_{2,3}+t_{3,2}+t_{3,3}+t_{1,2}+t_{2,1}+t_{2,2})$ from both sides of the inequality one can see that:

$$-\frac{1}{3}\cdot \sum_{\substack{i,j\geqslant 0\\ i+j\geqslant 2}} t_{i,j} \geqslant 2n+5-(k^2+k)-4\cdot \sum_{\substack{i,j\geqslant 1\\ |i-j|\leqslant 1\\ 2\leqslant i+j\leqslant 6}} t_{i,j}$$

$$\sum_{\substack{i,j\geqslant 1\\ |i-j|\leqslant 1\\ 2\leqslant i+j\leqslant 6}} t_{i,j} \geqslant \frac{t+6n+15-3(k^2+k)}{12}.$$

So we have now proven the following:

Theorem 4.2.13. *Let G and R each be a finite set of green or red points, respectively, in $\mathbb{R}^2$ such that $|G| = n$, $|R| = n-k$, $G\cap R = \emptyset$, and the points of $G\cup R$ are not all collinear. Let t be the total number of lines determined by $G\cup R$. The number of equichromatic lines determined by at most six points is at least $\left(t+6n+15-3k(k+1)\right)/12$.*

By letting $k = 0$, Theorem 4.2.13 becomes the corrected result for Pach and Pinchasi's. The fact that the lines counted are equichromatic, instead of the more general bichromatic, comes for free.

4.3 Lines in $\mathbb{C}^2$

4.3.1 A Lower Bound For Bichromatic Lines

In section 4.2.1, a lower bound on the number of equichromatic lines in $\mathbb{R}^2$ was shown to be $\frac{1}{4}(t + 2n + 3 - (k^2 + k))$. Below, a nearly equivalent lower bound is demonstrated for bichromatic lines in $\mathbb{C}^2$, assuming no $2n - k - 2$ points are collinear (note that $|G \cup R| = 2n - k$).

Hirzebruch's Improved Inequality (4.6) can be rewritten as (assuming n points):

$$9 \cdot \sum_{k \geqslant 2} t_k \geqslant n + 2 \cdot \sum_{k \geqslant 2} k t_k + 4t_2 + \frac{9}{4} t_3 + t_4.$$

In our context, this would be:

$$9 \cdot \sum_{\substack{i,j \geqslant 0 \\ i+j \geqslant 2}} t_{i,j} \geqslant 2n - k + 2 \cdot \sum_{\substack{i,j \geqslant 0 \\ i+j \geqslant 2}} (i + j) \cdot t_{i,j} + 4t_2 + \frac{9}{4} t_3 + t_4.$$

Using our identity (4.3):

$$9 \cdot \sum_{\substack{i,j \geqslant 0 \\ i+j \geqslant 2}} t_{i,j} - 6n + 2(k^2 + k) \geqslant 2 \cdot \sum_{\substack{i,j \geqslant 0 \\ i+j \geqslant 2}} (i - j)^2 \cdot t_{i,j} + 4t_2 + \frac{9}{4} t_3 + t_4.$$

On the right hand side of this inequality one can see that every monochromatic line is counted at least twelve times. Therefore,

$$m \leqslant \frac{3}{4} t - \frac{n}{2} + \frac{k^2 + k}{6}.$$

We get the following theorem:

Theorem 4.3.1. *Let G and R each be a finite set of green or red points, respectively, in $\mathbb{C}^2$ such that $|G| = n$, $|R| = n - k$, $G \cap R = \emptyset$, and no $2n - k - 2$ points of $G \cup R$ are collinear. Let t be the total number of lines determined by $G \cup R$. The number of bichromatic lines is at least* $(3t + 6n - 2k(k + 1))/12$.

Similar attempts to place a lower bound for equichromatic lines (Q) in $\mathbb{C}^2$

yields only $|Q| \geqslant n$. We refer the reader to Lemmas 4.2.11 and 4.2.12 for the cases where there are $2n-k-2$ or $2n-k-1$ collinear points.

4.3.2 Bichromatic Lines Through At Most Six Points

Returning to the inequality derived in section 4.2.3, let $\epsilon = \frac{1}{3}$ in (4.9). The inequality becomes:

$$\begin{aligned}\tfrac{2}{3}(t_{2,0}+t_{0,2}) - \tfrac{10}{3}t_{1,1} + 5(t_{0,3}+t_{3,0}) \\ - 3(t_{1,2}+t_{2,1}) + 12(t_{0,4}+t_{4,0}) - 4t_{2,2} \\ + \sum_{\substack{i,j\geqslant 0\\ i+j\geqslant 5}} \left((i-j)^2 + \tfrac{1}{3}(2(i+j)-9) + i + j - 9\right) t_{i,j} \\ \leqslant -\frac{14n}{3} + k(k+\frac{7}{3}).\end{aligned}$$

Let $\gamma_{i,j}$ be the coefficent for $t_{i,j}$ given in the inequality above. Of all coefficients for the bichromatic lines containing fewer than six points, the smallest is $\gamma_{2,2} = -4$. Of all other coefficients (i.e. for lines either not bichromatic or containing more than six points), the smallest are $\gamma_{0,2} = \gamma_{3,4} = \gamma_{4,3} = \frac{2}{3}$.

By adding $\frac{14}{3} \cdot \sum_{\substack{i,j\geqslant 1\\ 2\leqslant i+j\leqslant 6}} t_{i,j}$ to both sides and rearranging, one can see that:

$$\frac{2t}{3} + \frac{14n}{3} - k(k+\frac{7}{3}) \leqslant \frac{14}{3} \cdot \sum_{\substack{i,j\geqslant 1\\ 2\leqslant i+j\leqslant 6}} t_{i,j}$$

$$\sum_{\substack{i,j\geqslant 1\\ 2\leqslant i+j\leqslant 6}} t_{i,j} \geqslant \frac{t}{7} + n - \frac{k(3k+7)}{14}.$$

We get the following theorem:

Theorem 4.3.2. *Let G and R each be a finite set of green or red points, respectively, in $\mathbb{C}^2$ such that $|G| = n$, $|R| = n-k$, $G \cap R = \emptyset$, and no $2n-k-2$ points of $G \cup R$ are collinear. Let t be the total number of lines determined by $G \cup R$. The number of bichromatic lines determined by at most six points is at least* $(2t + 14n - k(3k+7))/14$.

We remark that this result is of course also true in $\mathbb{R}^2$, and is better than the lower bound found by Pach and Pinchasi. Again, we refer the reader to Lemmas 4.2.11 and 4.2.12 for the cases where there are $2n-k-2$ or $2n-k-1$ collinear points.

4.3.3 Lower Bound On Total Number of Lines[1]

In Section 4.2.2, we utilized a result from a Kelly-Moser paper [25] to prove a conjecture of Kleitman and Pinchasi. This result of Kelly and Moser (called "Theorem 4.1" in their paper) showed a lower bound, assuming certain conditions were met, on the total number of lines determined by an arrangement of points. Since their result was derived from Euler's Polyhedral Formula, it does not apply to the complex plane. Below, an analogous result is derived and is valid for $\mathbb{C}^2$.

Lemma 4.3.3. *Given r points on a line, L, and s points not on L, in $\mathbb{C}^2$ at least $1+rs-s(s-1)/2$ lines, including L, are formed.*

Proof. Using induction on s. If $s=1$, then there are $r+1$ lines and the lemma is true.

Suppose the lemma is true for s. A new point off of L will determine r lines, of which at most s already exist. Hence, we have at least

$$(1+rs-s(s-1)/2)+(r-s) = 1+r(s+1)-(s+1)s/2$$

lines. The lemma follows by induction. □

The lemma above is quite strong, but requires exactly "r points on a line, L, and s points not on L". To get the desired result, we must extend this to point sets where *at most* $n-k$ are collinear.

We now alter our terms. Instead of r and s, we will refer to a set of n points with at most $n-k$ collinear. Furthermore, we define the expression from

[1]The results of this section should primarily be attributed to my advisor, George Purdy.

the lemma above to be a function of k (with n being considered constant), i.e. $f(k) \stackrel{\text{def}}{=} k(n-k) - \frac{k(k-1)}{2} + 1$.

Theorem 4.3.4. *Given a finite set of points, G, in $\mathbb{C}^2$ such that $|G| = n$, $k \geqslant 2$ and at most $n-k$ of the points are collinear. Let t be the total number of lines determined by G. If,*

$$n \geqslant (3(16k^2 - 23k + 9) - 1)/2 \tag{4.10}$$

then,

$$t \geqslant f(k) = kn - \frac{(3k+2)(k-1)}{2}.$$

Proof. Let r_i denote the number of points that are incident to precisely i lines. (Note that we only consider lines determined by the set of points.) We assume that the points are not all collinear, so $r_1 = 0$. Fix $a > k$ to be a number determined later.

Case 1: Suppose $r_2 + r_3 + \ldots + r_{a+1} \geqslant 2$.

There are two points, P and Q, each incident with at most $a+1$ lines. Let L be the line PQ. There are at most a^2 points not on L, since neither P nor Q can be incident to a line, other than L, with more than $a+1$ points. Let $n-x$ be the exact number of points on L. Now consider the function $f(x)$ defined above. The first and second derivative of this function are $f'(x) = -3x + n + \frac{1}{2}$, and $f''(x) = -3$ (i.e $f(x)$ is concave down). Since $k \leqslant x \leqslant a^2$, we know $t \geqslant f(x) \geqslant min\{f(k), f(a^2)\}$.

We are going to choose n so that $f(k) \leqslant f(a^2)$, so that $t \geqslant f(k)$ and the theorem holds. (Note the ordering of values: $a^2 > a > k \geqslant 2$.) Thus we want:

$$k(n-k) - \frac{k(k-1)}{2} + 1 \leqslant a^2(n-a^2) - \frac{a^2(a^2-1)}{2} + 1.$$

And hence we want:

$$n(a^2 - k) \geqslant \frac{3(a^4 - k^2)}{2} - \frac{a^2 - k}{2}.$$

Dividing both sides by $a^2 - k > 0$, we obtain:

$$n \geqslant \frac{3(a^2 + k) - 1}{2}. \tag{4.11}$$

Case 2: Suppose $r_2 + r_3 + \ldots + r_{a+1} \leqslant 1$.

By Hirzebruch's first inequality (4.5), we know:

$$\begin{aligned} t_2 + t_3 + 4t &\geqslant 4t + n + t_5 + 2t_6 + 3t_7 + \ldots \\ &\geqslant n + 4t_2 + 4t_3 + 4t_4 + 5t_5 + 6t_6 + \ldots. \end{aligned}$$

Hence,

$$\begin{aligned} 4t &\geqslant n + 3t_2 + 3t_3 + 4t_4 + 5t_5 + \ldots \\ &\geqslant n + \sum_{i \geqslant 2} i \cdot t_i = n + \sum_{i \geqslant 2} i \cdot r_i \\ &\geqslant n + 2 + (a+2)(-1 + (r_2 + r_3 + \ldots + r_{a+1}) + r_{a+2} + \ldots) \\ &= n + 2 + (a+2)(n-1). \end{aligned}$$

We need to choose a so that $t \geqslant f(k) = k(n-k) - \frac{k(k-1)}{2} + 1$. We choose $a = 4k - 3$, so that $4t \geqslant 2 + (4k-1)(n-1) + n = 4kn - 4k + 3$, and so $t > kn - k > f(k)$. Combining this with (4.11) gives us (4.10).

□

4.3.4 The Kleitman-Pinchasi Conjecture Revisited

In Section 4.3.2 we proved a lower bound on the number of bichromatic lines passing through at most six points in $\mathbb{C}^2$. This result will now be used in conjunction with Theorem 4.3.4 to extend the proof of the Kleitman-Pinchasi conjecture to the complex plane.

The following lemmas, the same as used in Section 4.2.2, are also true in the complex plane ($\mathbb{C}^2$). (Note that we are now using the following context: Let G and R each be a finite set of green or red points, respectively, in $\mathbb{C}^2$ such that $|G| = n$, $|R| = n - k$, $k \in \{0, 1\}$, and $G \cap R = \emptyset$.)

Lemma 4.3.5. *If neither color class is collinear, and $2n - k - 2$ points are incident to one line then the number of equichromatic lines determined by at most three points is at least $2n - k - 1$.*

Lemma 4.3.6. *If neither color class is collinear, and $2n - k - 3$ points are incident to one line then the number of equichromatic lines determined by at most four points is at least $3n - k - 4$.*

With little effort, one can see from Theorem 4.3.2 that whenever $t \geq 7n - 7$, the number of bichromatic lines through no more than six points in $\mathbb{C}^2$ is greater than $2n - k - 1$. Using Theorem 4.3.4, we have the following lemma:

Lemma 4.3.7. *If $n \geqslant 130$ and no more than $2n - k - 4$ points are incident to one line, then $t \geqslant 8n - 4k - 21$.*

By combining Lemmas 4.3.5, 4.3.6, and 4.3.7 with Theorem 4.3.2 we get the following theorem:

Theorem 4.3.8. *Let G and R each be a finite set of green or red points, respectively, in $\mathbb{C}^2$ such that $|G| = n$, $|R| = n - k$, $k \in \{0, 1\}$, $G \cap R = \emptyset$, and neither color class is collinear. If $n \geqslant 130$, then the number of bichromatic lines determined by at most six points is at least $2n - k - 1 = |G| + |R| - 1$.*

Chapter 5

Conclusions and Future Work

5.1 Finding Ordinary or Monochromatic Intersection Points

It is conjectured that both $O(n \log n)$ algorithms presented in Chapter 3 are within a constant factor of the best upper bound for time.

It would be interesting to know whether an algorithm to find an ordinary intersection in an arrangement of pseudolines could also perform in time $O(n \log n)$. Likewise, it would be interesting to know whether an algorithm to find a monochromatic intersection in a bichromatic arrangement of pseudolines (or even lines) could perform in time $O(n \log n)$.

5.2 Bichromatic and Equichromatic Lines

For the convenience of the reader, Tables 5.1, 5.2, and 5.3 are a collection of the results from Chapter 4.

We ask whether one can prove a tight lower bound on the number of equichro-

Table 5.1: Best General Lower Bounds

	in $\mathbb{R}^2$	**in** $\mathbb{C}^2$ [a]
Equichromatic	$(t + 2n + 3 - k(k + 1))/4$	$(6n - k(k + 3))/4$
Bichromatic	$(t + 2n + 3 - k(k + 1))/4$ [b]	$(3t + 6n - 2k(k + 1))/12$

[a] Results in this column assume no $2n - k - 2$ points are collinear.
[b] For $k \geqslant 3$, $(3t + 6n - 2k(k + 1))/12$ is a better result.

Table 5.2: Best Equichromatic Lower Bounds

# of Points (at most)	**in** $\mathbb{R}^2$	**in** $\mathbb{C}^2$ [a]
4	$(2n + 6 - k(k + 1))/4$	N/A
5	$(6n - k(k + 3))/4$ [b]	$(6n - k(k + 3))/4$
6	$(t + 6n + 15 - 3k(k + 1))/12$	$(6n - k(k + 3))/4$

[a] Results in this column assume no $2n - k - 2$ points are collinear.
[b] Assumes no $2n - k - 2$ points are collinear.

Table 5.3: Best Bichromatic Lower Bounds

# of Points (at most)	**in** $\mathbb{R}^2$	**in** $\mathbb{C}^2$ [a]
4	$(2n + 6 - k(k + 1))/4$	N/A
5	$(6n - k(k + 3))/4$ [b]	$(6n - k(k + 3))/4$
6	$(2t + 14n - k(3k + 7))/14$ [c]	$(2t + 14n - k(3k + 7))/14$

[a] Results in this column assume no $2n - k - 2$ points are collinear.
[b] Assumes no $2n - k - 2$ points are collinear.
[c] Assumes no $2n - k - 2$ points are collinear.

matic or bichromatic lines determined by at most four points in $\mathbb{C}^2$.

Let t be the total number of lines determined by a point set. We conjecture that there exists an $\Omega(t)$ lower bound on the number of equichromatic lines in $\mathbb{C}^2$. We also conjecture that there exists a lower bound on the number of bichromatic lines determined by points in $\mathbb{C}^2$ (or $\mathbb{R}^2$) that is asymptotic to $t/2$. (The example of $|G| = |R| = n$ with points in general position gives $\binom{n}{2} + \binom{n}{2} = n^2 - n$ monochromatic lines and n^2 bichromatic lines, so $\sim t/2$ cannot be made stronger.)

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